

Crossed modules for Lie 2-algebras

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Abstract

The notion of crossed modules for Lie 2-algebras is introduced. We show that, associated to such a crossed module, there is a strict Lie 3-algebra structure on its mapping cone complex and a strict Lie 2-algebra structure on its derivations. Finally, we classify strong crossed modules by means of the third cohomology group of Lie 2-algebras.

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1 Introduction

Crossed modules of Lie algebras first appeared in the work of Gerstenhaber ([11]), which can be classified by use of the third cohomology group of Lie algebras as follows: for a crossed module of Lie algebras $\varphi : \mathfrak{m} \longrightarrow \mathfrak{g}$, there exists a four term exact sequence

$$0 \longrightarrow \mathbb{V} \xrightarrow{i} \mathfrak{m} \xrightarrow{\varphi} \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \longrightarrow 0,$$

⁰*Keyword:* crossed module, Lie 2-algebra, derivations.

where the cokernel \mathfrak{h} is a Lie algebra and the kernel \mathbb{V} is an \mathfrak{h} -module induced by the action of \mathfrak{g} on \mathfrak{m} . Denote by $\text{crmod}(\mathfrak{h}, \mathbb{V})$ the set of equivalence classes of crossed modules with fixed kernel \mathbb{V} , cokernel \mathfrak{h} and action. Gerstenhaber proved that there is a bijection between $\text{crmod}(\mathfrak{h}, \mathbb{V})$ and $H^3(\mathfrak{h}, \mathbb{V})$. See also [24] for more details and [3, 4] for other algebraic structures.

Lie algebras can be categorified to Lie 2-algebras. For a good introduction on this subject see [1, 14, 6]. A Lie 2-algebra is a 2-vector space equipped with a skew-symmetric bilinear functor, such that the Jacobi identity is controlled by a natural isomorphism, which satisfies the coherence law of its own. It is well-known that the notion of Lie 2-algebras is equivalent to that of 2-term L_∞ -algebras and the category of strict Lie 2-algebras is isomorphic to the category of crossed modules of Lie algebras ([1]).

The cohomology of L_∞ -algebras and A_∞ -algebras was studied in [14, 22, 16] and in [18] for a more general theory. While the cohomology of Lie 2-algebras was formulated in [2] for the strict case to characterize strict Lie 2-bialgebras and in [15] for the general case to depict the deformation of Lie 2-algebras. For the cohomology of Lie 2-groups, see [12]. In this paper, we propose the notion of crossed modules of Lie 2-algebras (Definition 3.2) and classify strong crossed modules via the third cohomology group of Lie 2-algebras (Theorem 5.10).

Moreover, given a crossed module of Lie 2-algebras $(\mathfrak{m}, \mathfrak{g}, \phi, \varphi, \sigma)$, there is a strict Lie 3-algebra ($l_4 = 0$) structure on its mapping cone complex: $\mathfrak{m}_1 \longrightarrow \mathfrak{g}_1 \oplus \mathfrak{m}_0 \longrightarrow \mathfrak{g}_0$ (Theorem 3.5). Also, we obtain a strict Lie 2-algebra structure on derivations $\text{Der}(\mathfrak{g}, \mathfrak{m}) : \text{Hom}(\mathfrak{g}_0, \mathfrak{m}_1) \xrightarrow{-D} \text{Der}_0(\mathfrak{g}, \mathfrak{m})$ (Theorem 4.4), where $\text{Der}_0(\mathfrak{g}, \mathfrak{m})$ is the set of 1-cocycles and D is the Lie 2-algebra coboundary operator. Moreover, we get a Lie algebra structure on the first cohomology group $H^1(\mathfrak{g}, \mathfrak{m})$.

This paper is organized as follows: In Section 2, we sketch some background on Lie 2-algebras, including basic definitions, the cohomology theory and the derivations of Lie 2-algebras. In Section 3, we introduce the notion of crossed modules of Lie 2-algebras with some examples and demonstrate that there is a strict Lie 3-algebra on the mapping cone complex. In Section 4, we provide a Lie algebra structure on the set of 1-cocycles. Then we prove that for a crossed module, there exists a strict Lie 2-algebra structure on its derivations $\text{Der}(\mathfrak{g}, \mathfrak{m})$ and a Lie algebra structure on $H^1(\mathfrak{g}, \mathfrak{m})$. Section 5 is concerned about the classification of strong crossed modules using the third cohomology group.

Acknowledgement: We would like to thank M. Markl for his useful comments on free Lie 2-algebras.

2 Background on Lie 2-algebras

2.1 Basic notions

L_∞ -algebras, also called strongly homotopy Lie algebras, were introduced by Drinfeld and Stasheff as a model for “Lie algebras that satisfy Jacobi identity up to all higher homotopies”. The following definition of L_∞ -structure was formulated by Stasheff in 1985. See [23].

Definition 2.1. *An L_∞ -algebra is a graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots$ equipped with a system $\{l_k \mid 1 \leq k < \infty\}$ of linear maps $l_k : \wedge^k \mathfrak{g} \longrightarrow \mathfrak{g}$ with degree $\deg(l_k) = k - 2$, where the exterior powers are interpreted in the graded sense and the following relation with Koszul sign “Ksgn” is satisfied for all $n \geq 0$:*

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \sum_{\sigma} \text{sgn}(\sigma) K \text{sgn}(\sigma) l_j(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0, \quad (1)$$

where the summation is taken over all $(i, n - i)$ -unshuffles with $i \geq 1$.

Usually, an n -term L_∞ -algebra (i.e., $l_i = 0, i \geq n+2$) is called a Lie n -algebra. In particular, if $l_{n+1} = 0$, it is called a strict Lie n -algebra. Next we focus on the case of $n = 2$.

Definition 2.2. Let $(\mathfrak{g}, d, l_2, l_3)$ and $(\mathfrak{g}', d', l'_2, l'_3)$ be Lie 2-algebras. A **Lie 2-algebra homomorphism** $\varphi : \mathfrak{g} \longrightarrow \mathfrak{g}'$ consists of

- two linear maps $\varphi_0 : \mathfrak{g}_0 \longrightarrow \mathfrak{g}'_0$ and $\varphi_1 : \mathfrak{g}_1 \longrightarrow \mathfrak{g}'_1$,
- one bilinear map $\varphi_2 : \mathfrak{g}_0 \wedge \mathfrak{g}_0 \longrightarrow \mathfrak{g}'_1$,

such that the following equalities hold for all $x, y, z \in \mathfrak{g}_0, a \in \mathfrak{g}_1$:

- $d' \circ \varphi_1 = \varphi_0 \circ d$,
- $\varphi_0 l_2(x, y) - l'_2(\varphi_0(x), \varphi_0(y)) = d' \varphi_2(x, y)$,
- $\varphi_1 l_2(x, a) - l'_2(\varphi_0(x), \varphi_1(a)) = \varphi_2(x, da)$,
- $l'_2(\varphi_0(x), \varphi_2(y, z)) + c.p. + l'_3(\varphi_0(x), \varphi_0(y), \varphi_0(z)) = \varphi_2(l_2(x, y), z) + c.p. + \varphi_1(l_3(x, y, z))$,

where *c.p.* means cyclic permutation. It is called a **strong homomorphism** if $\varphi_2 = 0$.

Lemma 2.3. Let $(\mathfrak{g}, d, l_2, l_3)$ be a Lie 2-algebra and $\mathfrak{h} \subset \mathfrak{g}$ a 2-vector subspace. Then $\mathfrak{g}/\mathfrak{h}$ is a quotient Lie 2-algebra if and only if

$$l_2(\mathfrak{h} \wedge \mathfrak{g}) \subset \mathfrak{h}, \quad l_3(\mathfrak{h}_0 \wedge \mathfrak{g}_0 \wedge \mathfrak{g}_0) \subset \mathfrak{h}_1. \quad (2)$$

We call \mathfrak{h} satisfying condition (2) an **ideal** of \mathfrak{g} . In fact, the projection $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h}$ becomes a strong homomorphism. We now give an analogue of the fundamental theorem of algebras for Lie 2-algebras.

Proposition 2.4. Let $\varphi : \mathfrak{g} \longrightarrow \mathfrak{g}'$ be a Lie 2-algebra homomorphism. Then,

- (1) $\text{Im} \varphi = \text{Im} \varphi_0 \oplus \text{Im} \varphi_1$ is a Lie 2-subalgebra of \mathfrak{g}' if $\text{Im} \varphi_2 \subset \text{Im} \varphi_1$;
- (2) $\ker \varphi = \ker \varphi_0 \oplus \ker \varphi_1$ is an ideal of \mathfrak{g} if $\varphi_2(\ker \varphi_0 \wedge \mathfrak{g}_0) = 0$.

Moreover, the two Lie 2-algebras $\mathfrak{g}/\ker \varphi$ and $\text{Im} \varphi$ are isomorphic if the two conditions above are satisfied.

Proof. By Definition 2.2 and $\text{Im} \varphi_2 \subset \text{Im} \varphi_1$, it is direct to see that $\text{Im} \varphi$ is a 2-vector subspace of \mathfrak{g}' such that l'_2, l'_3 are closed on it. Namely, $\text{Im} \varphi$ is a Lie 2-subalgebra of \mathfrak{g}' .

Similarly, by the first three conditions of a Lie 2-algebra homomorphism and $\varphi_2(\ker \varphi_0 \wedge \mathfrak{g}_0) = 0$, we get $l_2(\ker \varphi_0 \wedge \mathfrak{g}) \subset \ker \varphi$. Coupled with the last condition, we further obtain that $l_3(\ker \varphi_0 \wedge \mathfrak{g}_0 \wedge \mathfrak{g}_0) \subset \ker \varphi_1$. That is, $\ker \varphi$ is an ideal of \mathfrak{g} . The remaining result is immediate. ■

2.2 Cohomology

Given a \mathfrak{g} -module $\mathbb{V} : V_1 \xrightarrow{\partial} V_0$ with a Lie 2-algebra homomorphism $\phi : \mathfrak{g} \rightarrow \text{End}(\mathbb{V})$, ϕ is called an **action** of \mathfrak{g} on \mathbb{V} and denoted by:

$$x \triangleright u = \phi(x)u, \quad (x, y) \triangleright u = \phi_2(x, y)u, \quad \forall x, y \in \mathfrak{g}, u \in \mathbb{V}.$$

The cohomology group of a Lie 2-algebra $(\mathfrak{g}, d, l_2, l_3)$ comes from the generalized Chevalley-Eilenberg complex as follows:

$$\begin{aligned}
& \text{degree } -1 : V_1 \xrightarrow{D} \\
& \text{degree } 0 : V_0 \oplus \text{Hom}(\mathfrak{g}_0, V_1) \xrightarrow{D} \\
& \text{degree } 1 : \text{Hom}(\mathfrak{g}_0, V_0) \oplus \text{Hom}(\mathfrak{g}_1, V_1) \oplus \text{Hom}(\wedge^2 \mathfrak{g}_0, V_1) \xrightarrow{D} \\
& \text{degree } 2 : \text{Hom}(\mathfrak{g}_1, V_0) \oplus \text{Hom}(\wedge^2 \mathfrak{g}_0, V_0) \oplus \text{Hom}(\mathfrak{g}_0 \wedge \mathfrak{g}_1, V_1) \oplus \text{Hom}(\wedge^3 \mathfrak{g}_0, V_1) \xrightarrow{D} \\
& \xrightarrow{D} \dots
\end{aligned} \tag{3}$$

Denote by $C^i(\mathfrak{g}, \mathbb{V})$ the set of i-cochains. The coboundary operator D can be decomposed as:

$$D = \hat{d} + \hat{\partial} + d_\phi^{(1,0)} + d_\phi^{(0,1)} + d_{\phi_2} + d_{l_3},$$

where, for $s = 0, 1$,

$$\begin{aligned}
\hat{d} : \text{Hom}(\wedge^p \mathfrak{g}_0 \wedge \odot^q \mathfrak{g}_1, V_s) &\longrightarrow \text{Hom}(\wedge^{p-1} \mathfrak{g}_0 \wedge \odot^{q+1} \mathfrak{g}_1, V_s), \\
\hat{\partial} : \text{Hom}(\wedge^p \mathfrak{g}_0 \wedge \odot^q \mathfrak{g}_1, V_1) &\longrightarrow \text{Hom}(\wedge^p \mathfrak{g}_0 \wedge \odot^q \mathfrak{g}_1, V_0), \\
d_\phi^{(1,0)} : \text{Hom}(\wedge^p \mathfrak{g}_0 \wedge \odot^q \mathfrak{g}_1, V_s) &\longrightarrow \text{Hom}(\wedge^{p+1} \mathfrak{g}_0 \wedge \odot^q \mathfrak{g}_1, V_s), \\
d_\phi^{(0,1)} : \text{Hom}(\wedge^p \mathfrak{g}_0 \wedge \odot^q \mathfrak{g}_1, V_0) &\longrightarrow \text{Hom}(\wedge^p \mathfrak{g}_0 \wedge \odot^{q+1} \mathfrak{g}_1, V_1), \\
d_{\phi_2} : \text{Hom}(\wedge^p \mathfrak{g}_0 \wedge \odot^q \mathfrak{g}_1, V_0) &\longrightarrow \text{Hom}(\wedge^{p+2} \mathfrak{g}_0 \wedge \odot^q \mathfrak{g}_1, V_1), \\
d_{l_3} : \text{Hom}(\wedge^p \mathfrak{g}_0 \wedge \odot^q \mathfrak{g}_1, V_s) &\longrightarrow \text{Hom}(\wedge^{p+3} \mathfrak{g}_0 \wedge \odot^{q-1} \mathfrak{g}_1, V_s).
\end{aligned}$$

More concretely, for any $x_i \in \mathfrak{g}_0, a_i \in \mathfrak{g}_1, i \in \mathbb{N}$,

$$\begin{aligned}
\hat{d}f(x_1, \dots, x_{p-1}, a_1, \dots, a_{q+1}) &= (-1)^p (f(x_1, \dots, x_{p-1}, dh_1, h_2, \dots, h_{q+1}) + c.p.(a_1, \dots, a_{q+1})), \\
\hat{\partial}f &= (-1)^{p+2q} \partial \circ f,
\end{aligned}$$

$$\begin{aligned}
d_\phi^{(1,0)} f(x_1, \dots, x_{p+1}, a_1, \dots, a_q) &= \sum_{i=1}^{p+1} (-1)^{i+1} x_i \triangleright f(x_1, \dots, \widehat{x_i}, \dots, x_{p+1}, a_1, \dots, a_q) \\
&\quad + \sum_{i < j} (-1)^{i+j} f([x_i, x_j], x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{p+1}, a_1, \dots, a_q) \\
&\quad + \sum_{i,j} (-1)^i f(x_1, \dots, \widehat{x_i}, \dots, x_{p+1}, a_1, \dots, [x_i, a_j], \dots, a_q), \\
d_\phi^{(0,1)} f(x_1, \dots, x_p, a_1, \dots, a_{q+1}) &= \sum_{i=1}^{q+1} (-1)^p a_i \triangleright f(x_1, \dots, x_p, a_1, \dots, \widehat{a_i}, \dots, a_{q+1}), \\
d_{\phi_2} f(x_1, \dots, x_{p+2}, a_1, \dots, a_q) &= \sum_{\sigma} (-1)^{p+2q} (-1)^\sigma (x_{\sigma(1)}, x_{\sigma(2)}) \triangleright f(x_{\sigma(3)}, \dots, x_{\sigma(p+2)}, a_1, \dots, a_q), \\
d_{l_3} f(x_1, \dots, x_{p+3}, a_1, \dots, a_{q-1}) &= \sum_{\tau} -(-1)^\tau f(x_{\tau(4)}, \dots, x_{\tau(p+3)}, a_1, \dots, a_{q-1}, l_3(x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)})).
\end{aligned}$$

where σ and τ are taken over all $(2, p)$ -unshuffles and $(3, p)$ -unshuffles respectively.

By direct calculations, we get the expressions for 1-cocycles and 1-coboundaries.

Lemma 2.5. *Let $X + l_X \in C^1(\mathfrak{g}, \mathbb{V})$, where $X = (X_0, X_1) \in \text{Hom}(\mathfrak{g}_0, V_0) \oplus \text{Hom}(\mathfrak{g}_1, V_1)$ and $l_X \in \text{Hom}(\wedge^2 \mathfrak{g}_0, V_1)$. Then*

(1) $D(X + l_X) = 0$ if and only if

$$X_0 \circ d = \partial \circ X_1, \tag{4}$$

$$\partial l_X(x, y) = X[x, y] + y \triangleright Xx - x \triangleright Xy, \tag{5}$$

$$l_X(x, da) = X[x, a] + a \triangleright Xx - x \triangleright Xa, \tag{6}$$

$$Xl_3(x, y, z) = l_X(x, [y, z]) + x \triangleright l_X(y, z) - (y, z) \triangleright Xx + c.p.(x, y, z). \tag{7}$$

(2) $\exists u + \Theta \in V_0 \oplus \text{Hom}(\mathfrak{g}_0, V_1) = C^0(\mathfrak{g}, \mathbb{V})$, s.t. $X + l_X = D(u + \Theta)$ if and only if

$$X(x + a) = x \triangleright u + a \triangleright u - \partial\Theta(x) - \Theta(da), \quad (8)$$

$$l_X(x, y) = (x, y) \triangleright u + x \triangleright \Theta(y) - y \triangleright \Theta(x) - \Theta([x, y]). \quad (9)$$

For future references, we specify the 3-coboundaries. For a cochain $\lambda = \sum_{i=0}^3 \lambda_i \in C^2(\mathfrak{g}, \mathbb{V})$, where

$$\lambda_0 \in \text{Hom}(\mathfrak{g}_1, V_0), \lambda_1 \in \text{Hom}(\wedge^2 \mathfrak{g}_0, V_0), \lambda_2 \in \text{Hom}(\mathfrak{g}_0 \wedge \mathfrak{g}_1, V_1), \lambda_3 \in \text{Hom}(\wedge^3 \mathfrak{g}_0, V_1),$$

then $\theta = D\lambda$ has five components as follows:

$$\left\{ \begin{array}{ll} \theta_0 &= d_\phi^{(1,0)} \lambda_0 + \hat{d} \lambda_1 + \hat{\partial} \lambda_2 & \in \text{Hom}(\mathfrak{g}_0 \wedge \mathfrak{g}_1, V_0), \\ \theta_1 &= d_\phi^{(0,1)} \lambda_0 + \hat{d} \lambda_2 & \in \text{Hom}(\odot^2 \mathfrak{g}_1, V_1), \\ \theta_2 &= d_{l_3} \lambda_0 + d_\phi^{(1,0)} \lambda_1 + \hat{\partial} \lambda_3 & \in \text{Hom}(\wedge^3 \mathfrak{g}_0, V_0), \\ \theta_3 &= d_{\phi_2} \lambda_0 + d_\phi^{(0,1)} \lambda_1 + d_\phi^{(1,0)} \lambda_2 + \hat{d} \lambda_3 & \in \text{Hom}(\wedge^2 \mathfrak{g}_0 \wedge \mathfrak{g}_1, V_1), \\ \theta_4 &= d_{\phi_2} \lambda_1 + d_{l_3} \lambda_2 + d_\phi^{(1,0)} \lambda_3 & \in \text{Hom}(\wedge^4 \mathfrak{g}_0, V_1). \end{array} \right.$$

More precisely, for any $x, y, z, x_i \in \mathfrak{g}_0, a, b \in \mathfrak{g}_1$,

$$\left\{ \begin{array}{ll} \theta_0(x, a) &= x \triangleright \lambda_0(a) - \lambda_0[x, a] + \lambda_1(x, da) - \partial \lambda_2(x, a), \\ \theta_1(a, b) &= a \triangleright \lambda_0(b) + b \triangleright \lambda_0(a) - \lambda_2(da, b) - \lambda_2(db, a), \\ \theta_2(x, y, z) &= -\lambda_0 l_3(x, y, z) + (x \triangleright \lambda_1(y, z) - \lambda_1([x, y], z) + c.p.) - \partial \lambda_3(x, y, z), \\ \theta_3(x, y, a) &= (x, y) \triangleright \lambda_0(a) + a \triangleright \lambda_1(x, y) + x \triangleright \lambda_2(y, u) - y \triangleright \lambda_2(x, a) - \lambda_2([x, y], a) \\ &\quad - \lambda_2(y, [x, a]) + \lambda_2(x, [y, a]) - \lambda_3(x, y, da), \\ \theta_4(x_1, \dots, x_4) &= \sum_{\sigma} (-1)^{\sigma} (x_{\sigma_1}, x_{\sigma_2}) \triangleright \lambda_1(x_{\sigma_3}, x_{\sigma_4}) - \sum_{\tau} (-1)^{\tau} \lambda_2(x_{\tau_4}, l_3(x_{\tau_1}, x_{\tau_2}, x_{\tau_3})) \\ &\quad + \sum_{i=1}^4 (-1)^{i+1} x_i \triangleright \lambda_3(x_1, \dots, \hat{x}_i, \dots, x_4) \\ &\quad + \sum_{i < j} (-1)^{i+j} \lambda_3([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_4). \end{array} \right. \quad (10)$$

2.3 Derivations

For a Lie 2-algebra $(\mathfrak{g}, d, [\cdot, \cdot], l_3)$, there is a natural **adjoint action** ad of \mathfrak{g} on itself given by

$$ad(x) = [x, \cdot], \quad ad_2(y, z) = -l_3(y, z, \cdot), \quad \forall x, y, z \in \mathfrak{g}.$$

To propose the crossed module of Lie 2-algebras below, we review the notion of derivations of a Lie 2-algebra $\text{Der}(\mathfrak{g})$, which was proved to be a strict Lie 2-algebra in [5].

Let $(\mathfrak{g}, d, [\cdot, \cdot], l_3)$ be a Lie 2-algebra. A derivation of degree 0 of \mathfrak{g} is a pair (X, l_X) , also denoted by $X + l_X$, where $X = (X_0, X_1) \in \text{Hom}(\mathfrak{g}_0, \mathfrak{g}_0) \oplus \text{Hom}(\mathfrak{g}_1, \mathfrak{g}_1)$ and $l_X : \mathfrak{g}_0 \wedge \mathfrak{g}_0 \rightarrow \mathfrak{g}_1$ is a linear map, such that for any $x, y, z \in \mathfrak{g}_0, a \in \mathfrak{g}_1$,

$$\left\{ \begin{array}{ll} d \circ X_1 &= X_0 \circ d, \\ dl_X(x, y) &= X[x, y] - [Xx, y] - [x, Xy], \\ l_X(x, da) &= X[x, a] - [Xx, a] - [x, Xa], \\ Xl_3(x, y, z) &= l_X(x, [y, z]) + [x, l_X(y, z)] + l_3(Xx, y, z) + c.p.(x, y, z). \end{array} \right. \quad (11)$$

Denote by $\text{Der}_0(\mathfrak{g})$ the set of derivations of degree 0 of \mathfrak{g} . Then one can define a 2-vector space as

$$\text{Der}(\mathfrak{g}) : \text{Der}_1(\mathfrak{g}) \triangleq \text{Hom}(\mathfrak{g}_0, \mathfrak{g}_1) \xrightarrow{\bar{d}} \text{Der}_0(\mathfrak{g}),$$

where \bar{d} is given by $\bar{d}(\Theta) = \delta(\Theta) + l_{\delta(\Theta)}$, in which $\delta(\Theta) = d \circ \Theta + \Theta \circ d$ and

$$l_{\delta(\Theta)}(x, y) = \Theta[x, y] - [x, \Theta y] - [\Theta x, y].$$

In addition, define $\{X + l_X, \Theta\} = [X, \Theta]$ and

$$\{X + l_X, Y + l_Y\} = [X, Y] + X \triangleright l_Y - Y \triangleright l_X, \quad (12)$$

where $[\cdot, \cdot]$ is the commutator bracket and

$$X \triangleright l_Y(x, y) = X l_Y(x, y) - l_Y(Xx, y) - l_Y(x, Xy).$$

Theorem 2.6. [5] *With notations above, $(\text{Der}(\mathfrak{g}), \{\cdot, \cdot\})$ is a strict Lie 2-algebra.*

Remark 2.7. *From the homological viewpoint, we discover an alternative description of $\text{Der}_0(\mathfrak{g})$ and \bar{d} . Comparing Equations (11) and \bar{d} with (4)-(7) and (9) respectively, we note that $\text{Der}_0(\mathfrak{g})$ is indeed the set of 1-cocycles of the Lie 2-algebra \mathfrak{g} with respect to the adjoint action and $\bar{d} = -D$, which D is the Lie 2-algebra coboundary operator.*

The adjoint action ad can be extended to a new Lie 2-algebra homomorphism $\overline{ad} : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$, where

$$\overline{ad}_0(x) = -D(x) = ad_0(x) + l_3(x, \cdot, \cdot), \quad \overline{ad}_1 = ad_1, \quad \overline{ad}_2 = ad_2, \quad \forall x \in \mathfrak{g}_0. \quad (13)$$

We conclude this section by exploring the derivations of a skeleton Lie 2-algebra, which turns out to have an explicit homological description.

Example 2.8. Let \mathfrak{g}_0 be an Lie algebra and V a \mathfrak{g}_0 -module. Given an Lie algebra 3-cocycle $l_3 \in C^3(\mathfrak{g}_0, V)$, we get a skeletal Lie 2-algebra $\mathfrak{g} = (V \xrightarrow{0} \mathfrak{g}_0, l_2, l_3)$, where l_2 is defined by

$$l_2^0(x, y) = [x, y]_{\mathfrak{g}_0}, \quad l_2^1(x, u) = x \triangleright u, \quad \forall x, y \in \mathfrak{g}_0, u \in V.$$

For $X = (X_0, X_1) \in \text{Hom}(\mathfrak{g}_0, \mathfrak{g}_0) \oplus \text{Hom}(V, V)$ and $l_X \in C^2(\mathfrak{g}_0, V)$, it is easy to check that

$$X + l_X \in \text{Der}_0(\mathfrak{g}) \iff \begin{cases} X & \in \text{Der}(\mathfrak{g}_0 \ltimes V), \\ \mathfrak{D}l_X & = [X, l_3], \end{cases}$$

where $\mathfrak{D} : C^k(\mathfrak{g}_0, V) \rightarrow C^{k+1}(\mathfrak{g}_0, V)$ is the Lie algebra coboundary operator, $\text{Der}(\mathfrak{g}_0 \ltimes V)$ is the Lie algebra of derivations of the semi-product Lie algebra $\mathfrak{g}_0 \ltimes V$ and the bracket $[\cdot, \cdot]$ is given by

$$[X, l_3](x, y, z) = X l_3(x, y, z) - l_3(X_0 x, y, z) - l_3(x, X_0 y, z) - l_3(x, y, X_0 z), \quad \forall x, y, z \in \mathfrak{g}_0.$$

In fact, such a bracket was used to introduce the notion of pre-Lie algebras by Gerstenhaber in [10]. In this case, we have $\text{Der}_1(\mathfrak{g}) = C^1(\mathfrak{g}_0, V)$ and the map $\bar{d} : \text{Der}_1(\mathfrak{g}) \rightarrow \text{Der}_0(\mathfrak{g})$ is given by $\bar{d}(\Theta) = 0 - \mathfrak{D}(\Theta)$.

3 Crossed modules of Lie 2-algebras

3.1 Definition of crossed modules

Let $(\mathfrak{m} : \mathfrak{m}_1 \xrightarrow{\bar{d}} \mathfrak{m}_0, \tilde{l}_2, \tilde{l}_3)$ and $(\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{d} \mathfrak{g}_0, l_2, l_3)$ be two Lie 2-algebras. We call \mathfrak{g} **acts on \mathfrak{m} by derivations** if there exists a Lie 2-algebra homomorphism $\phi : \mathfrak{g} \rightarrow \text{End}(\mathfrak{m})$ and a linear map $l_{\phi_0}(x) : \wedge^2 \mathfrak{m}_0 \rightarrow \mathfrak{m}_1$ such that $\phi_0(x) + l_{\phi_0}(x) \in \text{Der}_0(\mathfrak{m})$ and the map

$$(\phi_0 + l_{\phi_0}, \phi_1, \phi_2) : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{m})$$

is a Lie 2-algebra homomorphism. By abuse of notations, we denote by ϕ both the action and the action by derivations. Then we shall define a **crossed product** of \mathfrak{g} and \mathfrak{m} denoted by $\mathfrak{g} \triangleright_{\phi} \mathfrak{m}$, which is still a Lie 2-algebra depending on the following lemma.

Lemma 3.1. *Let ϕ be an action of \mathfrak{g} on \mathfrak{m} by derivations, then $\mathfrak{g} \triangleright_{\phi} \mathfrak{m} \triangleq (\mathfrak{g} \oplus \mathfrak{m}, L_1, [\cdot, \cdot], L_3)$ is a Lie 2-algebra with \mathfrak{g} as a lie 2-subalgebra and \mathfrak{m} as an ideal, where $L_1 = d + \tilde{d}$ and*

$$\left\{ \begin{array}{lcl} [x + \alpha, y + \beta] & = & l_2(x, y) + \tilde{l}_2(\alpha, \beta) + x \triangleright \beta - y \triangleright \alpha, \quad \forall x, y \in \mathfrak{g}, \forall \alpha, \beta \in \mathfrak{m}, \\ L_3(x + \alpha, y + \beta, z + \gamma) & = & l_3(x, y, z) + \tilde{l}_3(\alpha, \beta, \gamma) - (x, y) \triangleright \gamma - (y, z) \triangleright \alpha \\ & & - (z, x) \triangleright \beta + l_{\phi_0(x)}(\beta, \gamma) + l_{\phi_0(y)}(\gamma, \alpha) + l_{\phi_0(z)}(\alpha, \beta), \\ & & \forall x, y, z \in \mathfrak{g}_0, \forall \alpha, \beta, \gamma \in \mathfrak{m}_0. \end{array} \right.$$

Conversely, let (θ, L_1, L_2, L_3) be a Lie 2-algebra which can be split into the direct sum of a Lie 2-subalgebra \mathfrak{g} and an ideal \mathfrak{m} , then there exists an action ϕ of \mathfrak{g} on \mathfrak{m} by derivations such that $\theta = \mathfrak{g} \triangleright_{\phi} \mathfrak{m}$, where $\phi : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{m})$ is defined by

$$\left\{ \begin{array}{lcl} \phi_0(x) + l_{\phi_0(x)} & = & L_2(x, \cdot) + L_3(x, \cdot, \cdot), \quad \forall x \in \mathfrak{g}_0, \\ \phi_1(a) & = & L_2(a, \cdot), \quad \forall a \in \mathfrak{g}_1, \\ \phi_2(x, y) & = & -L_3(x, y, \cdot), \quad \forall x, y \in \mathfrak{g}_0. \end{array} \right.$$

Proof. We merely provide the proof of the coherence law of L_2, L_3 . Denote by $[\cdot, \cdot]$ both l_2 and \tilde{l}_2 if there is no risk of confusion. Firstly, for four elements in \mathfrak{g}_0 or four elements in \mathfrak{m}_0 , it holds since \mathfrak{g} and \mathfrak{m} are Lie 2-algebras and L_2, L_3 reserve their brackets. For any $x, y, z \in \mathfrak{g}_0, \alpha, \beta, \gamma \in \mathfrak{m}_0$, we have

$$\begin{aligned} & [L_3(x, \alpha, \beta), \gamma] + L_3(x, [\alpha, \beta], \gamma) - L_3([x, \alpha], \beta, \gamma) + c.p.(\alpha, \beta, \gamma) - [L_3(\alpha, \beta, \gamma), x] \\ = & [l_{\phi_0(x)}(\alpha, \beta), \gamma] + l_{\phi_0(x)}([\alpha, \beta], \gamma) - \tilde{l}_3(x \triangleright \alpha, \beta, \gamma) + c.p.(\alpha, \beta, \gamma) + x \triangleright \tilde{l}_3(\alpha, \beta, \gamma), \end{aligned}$$

which vanishes since $\phi_0(x) + l_{\phi_0(x)} \in \text{Der}_0(\mathfrak{m})$. Next, making use of the fact ϕ is an action by derivations coupled with (12), we have

$$l_{\phi_0[x, y]} - (\phi_0(x) \triangleright l_{\phi_0(y)} - \phi_0(y) \triangleright l_{\phi_0(x)}) = l_{\delta\phi_2(x, y)}.$$

Hence,

$$\begin{aligned} & -[L_3(y, \alpha, \beta), x] + L_3([x, \alpha], y, \beta) - L_3([x, \beta], y, \alpha) - c.p.(x, y) \\ & + [L_3(x, y, \alpha), \beta] - [L_3(\beta, x, y), \alpha] - L_3([\alpha, \beta], x, y) - L_3([x, y], \alpha, \beta) \\ = & x \triangleright l_{\phi_0(y)}(\alpha, \beta) - l_{\phi_0(y)}(x \triangleright \alpha, \beta) - l_{\phi_0(y)}(\alpha, x \triangleright \beta) - c.p.(x, y) \\ & - [(x, y) \triangleright \alpha, \beta] + [(x, y) \triangleright \beta, \alpha] + (x, y) \triangleright [\alpha, \beta] - l_{\phi_0[x, y]}(\alpha, \beta) \\ = & (\phi_0(x) \triangleright l_{\phi_0(y)} - \phi_0(y) \triangleright l_{\phi_0(x)})(\alpha, \beta) + l_{\delta\phi_2(x, y)}(\alpha, \beta) - l_{\phi_0[x, y]}(\alpha, \beta) \\ = & 0. \end{aligned}$$

Now, it remains to show

$$\begin{aligned} & -[L_3(y, z, \alpha), x] - L_3([x, \alpha], y, z) - L_3([x, y], z, \alpha) + c.p.(x, y, z) + [L_3(x, y, z), \alpha] \\ = & -x \triangleright ((y, z) \triangleright \alpha) + (y, z) \triangleright (x \triangleright \alpha) + ([x, y], z) \triangleright \alpha + c.p.(x, y, z) + [L_3(x, y, z), \alpha] \\ = & ([\phi_2(y, z), \phi_0(x)] + \phi_2([x, y], z) + c.p.(x, y, z) + \phi_1(l_3(x, y, z)))\alpha \\ = & 0, \end{aligned}$$

where the last equality follows from that ϕ is a Lie 2-algebra homomorphism. This finishes the proof of $\mathfrak{g} \triangleright_{\phi} \mathfrak{m}$ is Lie 2-algebra. The remaining results are easy to get. ■

Definition 3.2. A **crossed module of Lie 2-algebras** is a quadruple $(\mathfrak{m}, \mathfrak{g}, \phi, \Pi)$, where $\mathfrak{m}, \mathfrak{g}$ are two Lie 2-algebras, ϕ is an action of \mathfrak{g} on \mathfrak{m} by derivations, and $\Pi : \mathfrak{g} \triangleright_{\phi} \mathfrak{m} \rightarrow \mathfrak{g}$ is a Lie 2-algebra homomorphism, such that $\Pi|_{\mathfrak{g}} = \text{Id} = (\text{id}, \text{id}, 0)$ and

- (i) $\tilde{l}_2(\alpha, \beta) = \Pi(\alpha) \triangleright \beta, \quad \forall \alpha, \beta \in \mathfrak{m},$
- (ii) $\tilde{l}_3(\alpha, \beta, \gamma) = -(\Pi_0 \alpha, \Pi_0 \beta) \triangleright \gamma - \Pi_2(\Pi_0 \alpha, \beta) \triangleright \gamma, \quad \forall \alpha, \beta, \gamma \in \mathfrak{m}_0,$
- (iii) $l_{\phi_0(x)}(\beta, \gamma) = -(x, \Pi_0 \beta) \triangleright \gamma - \Pi_2(x, \beta) \triangleright \gamma, \quad \forall \beta, \gamma \in \mathfrak{m}_0, x \in \mathfrak{g}_0,$
- (iv) $\Pi_2(\alpha, \beta) = \Pi_2(\Pi_0 \alpha, \beta) = \Pi_2(\alpha, \Pi_0 \beta), \quad \forall \alpha, \beta \in \mathfrak{m}_0.$

In particular, it is called a **strong crossed module of Lie 2-algebras** if $\Pi_2 = 0$.

We drop the words ‘‘Lie 2-algebras’’ except when emphasis is needed. For a crossed module $(\mathfrak{m}, \mathfrak{g}, \phi, \Pi)$, decompose Π into

$$\Pi = (\Pi_0, \Pi_1, \Pi_2) = Id + \sigma + \varphi = ((id, \varphi_0), (id, \varphi_1), (0, \sigma, \varphi_2))$$

where $\varphi = \Pi|_{\mathfrak{m}}$ and $\sigma = \Pi_2|_{\mathfrak{g}_0 \wedge \mathfrak{m}_0}$. It is evident that $\varphi : \mathfrak{m} \rightarrow \mathfrak{g}$ is a Lie 2-algebra homomorphism. In the following, we always describe a crossed module as $(\mathfrak{m}, \mathfrak{g}, \phi, \varphi, \sigma)$. Then a strong crossed module means $\sigma = 0$ and is denoted by $(\mathfrak{m}, \mathfrak{g}, \phi, \varphi)$.

Definition 3.3. Let $(\mathfrak{m}, \mathfrak{g}, \phi, \varphi, \sigma)$ and $(\mathfrak{m}', \mathfrak{g}', \phi', \varphi', \sigma')$ be crossed modules, a **morphism of crossed modules** consists of two Lie 2-algebra homomorphisms $F : \mathfrak{m} \rightarrow \mathfrak{m}'$, $G : \mathfrak{g} \rightarrow \mathfrak{g}'$ and a linear map $\tau : \mathfrak{g}_0 \wedge \mathfrak{m}_0 \rightarrow \mathfrak{m}'_1$ such that $\varphi' \circ F = G \circ \varphi$ and

$$((G_0, F_0), (G_1, F_1), (G_2, \tau, F_2)) : \mathfrak{g} \triangleright_{\phi} \mathfrak{m} \rightarrow \mathfrak{g}' \triangleright_{\phi'} \mathfrak{m}'$$

is a homomorphism of Lie 2-algebras. If G_2, F_2, τ vanish, we call it a strong morphism.

Similar to the Lie algebra case, we have the following proposition.

Proposition 3.4. Let \mathfrak{g} be a Lie 2-algebra and \mathfrak{m} a \mathfrak{g} -module. Given a chain map $\varphi : \mathfrak{m} \rightarrow \mathfrak{g}$ and a map $\sigma : \mathfrak{g}_0 \wedge \mathfrak{m}_0 \rightarrow \mathfrak{g}_1$, satisfying that

$$\Pi \triangleq ((id, \varphi_0), (id, \varphi_1), (0, \sigma, 0)) : \mathfrak{g} \ltimes_{\phi} \mathfrak{m} \rightarrow \mathfrak{g}$$

is a Lie 2-algebra homomorphism and

- (1) $\varphi(\alpha) \triangleright \beta = -\varphi(\beta) \triangleright \alpha, \quad \forall \alpha, \beta \in \mathfrak{m},$
- (2) $(\varphi_0 \alpha, \varphi_0 \beta) \triangleright \gamma + \sigma(\varphi_0 \alpha, \beta) \triangleright \gamma = -(\varphi_0 \alpha, \varphi_0 \gamma) \triangleright \beta - \sigma(\varphi_0 \alpha, \gamma) \triangleright \beta, \quad \forall \alpha, \beta, \gamma \in \mathfrak{m}_0,$
- (3) $(x, \varphi_0 \beta) \triangleright \gamma + \sigma(x, \beta) \triangleright \gamma = -(x, \varphi_0 \gamma) \triangleright \beta - \sigma(x, \gamma) \triangleright \beta, \quad \forall \beta, \gamma \in \mathfrak{m}_0, x \in \mathfrak{g}_0,$
- (4) $\sigma(\varphi_0 \alpha, \beta) = \sigma(\alpha, \varphi_0 \beta), \quad \forall \alpha, \beta \in \mathfrak{m}_0,$

then, there exists a unique Lie 2-algebra structure on \mathfrak{m} , linear maps $l_{\phi_0(x)} : \wedge^2 \mathfrak{m}_0 \rightarrow \mathfrak{m}_1$ and $\varphi_2 : \wedge^2 \mathfrak{m}_0 \rightarrow \mathfrak{g}_1$ such that $(\mathfrak{m}, \mathfrak{g}, \hat{\phi}, \hat{\varphi}, \sigma)$ is a crossed module, where $\hat{\phi} = (\phi_0 + l_{\phi_0}, \phi_1, \phi_2)$ and $\hat{\varphi} = (\varphi_0, \varphi_1, \varphi_2)$.

Proof. Define $\tilde{l}_2, \tilde{l}_3, l_{\phi_0(x)}, \varphi_2$ on \mathfrak{m} by the right hand sides of equalities (i)-(iv) of Definition 3.2. Then, by direct verification, we obtain that \mathfrak{m} with \tilde{l}_2, \tilde{l}_3 is a Lie 2-algebra and $\hat{\phi} : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{m})$ is a Lie 2-algebra homomorphism. Moreover, $\hat{\Pi} = ((id, \varphi_0), (id, \varphi_1), (0, \sigma, \varphi_2)) : \mathfrak{g} \triangleright_{\hat{\phi}} \mathfrak{m} \rightarrow \mathfrak{g}$ is a Lie 2-algebra homomorphism. Thus, we get a crossed module of Lie 2-algebras. ■

3.2 Lie 3-algebras associated to crossed modules

A crossed module $(\mathfrak{m}, \mathfrak{g}, \phi, \varphi, \sigma)$ corresponds to a square

$$\begin{array}{ccc} \mathfrak{m}_1 & \xrightarrow{\varphi_1} & \mathfrak{g}_1 \\ \tilde{d} \downarrow & & \downarrow d \\ \mathfrak{m}_0 & \xrightarrow{\varphi_0} & \mathfrak{g}_0. \end{array}$$

Consider its mapping cone complex ([13])

$$\mathfrak{V} : \mathfrak{m}_1 \xrightarrow{d_D} \mathfrak{g}_1 \oplus \mathfrak{m}_0 \xrightarrow{d_D} \mathfrak{g}_0, \quad (14)$$

where

$$d_D(\xi) = -\varphi_1 \xi + \tilde{d}\xi, \quad d_D(a + \alpha) = da + \varphi_0 \alpha, \quad \forall \xi \in \mathfrak{m}_1, a + \alpha \in \mathfrak{g}_1 \oplus \mathfrak{m}_0.$$

Then $d_D^2 = 0$ follows from the fact that $d \circ \varphi_1 = \varphi_0 \circ \tilde{d}$.

Define $\llbracket \cdot, \cdot \rrbracket$ and l^3 on it by: for any $x, y, z \in \mathfrak{g}_0$, $a, b \in \mathfrak{g}_1$, $\alpha, \beta \in \mathfrak{m}_0$ and $\xi \in \mathfrak{m}_1$,

$$\left\{ \begin{array}{ll} \llbracket x, y \rrbracket &= l_2(x, y), \\ \llbracket x, \xi \rrbracket &= -\llbracket \xi, x \rrbracket = x \triangleright \xi, \\ \llbracket a + \alpha, b + \beta \rrbracket &= a \triangleright \beta + b \triangleright \alpha, \\ \llbracket x, a + \alpha \rrbracket &= -\llbracket a + \alpha, x \rrbracket = l_2(x, a) - \sigma(x, \alpha) + x \triangleright \alpha, \\ l^3(x, y, z) &= l_3(x, y, z), \\ l^3(x, y, a + \alpha) &= -l^3(x, a + \alpha, y) = l^3(a + \alpha, x, y) = -(x, y) \triangleright \alpha. \end{array} \right.$$

Theorem 3.5. *With the above notations, $(\mathfrak{V}, d_D, \llbracket \cdot, \cdot \rrbracket, l^3)$ is a strict Lie 3-algebra.*

Proof. We need to verify all the conditions of Lie 3-algebras. Firstly, it is obvious that $\llbracket \cdot, \cdot \rrbracket$ and l^3 are antisymmetric in the graded sense. Then, it suffices to prove that equality (1) of Definition 2.1 holds for $1 \leq n \leq 5$.

- $n = 1$: (1) reduces to $d_D^2 = 0$, which has already been checked.
- $n = 2$: Condition (1) gives

$$d_D[x, y] = \llbracket d_D x, y \rrbracket + (-1)^{|x|} \llbracket x, d_D y \rrbracket, \quad \forall x, y \in \mathfrak{V},$$

which is equivalent to

$$\left\{ \begin{array}{ll} d_D \llbracket x, \xi \rrbracket &= \llbracket x, d_D \xi \rrbracket, \\ d_D \llbracket x, a + \alpha \rrbracket &= \llbracket x, d_D(a + \alpha) \rrbracket, \\ \llbracket d_D(a + \alpha), \xi \rrbracket &= \llbracket a + \alpha, d_D \xi \rrbracket, \\ d_D \llbracket a + \alpha, b + \beta \rrbracket &= \llbracket d_D(a + \alpha), b + \beta \rrbracket - \llbracket a + \alpha, d_D(b + \beta) \rrbracket. \end{array} \right.$$

The first three equations are easy to verify. As for the last one, by direct computations, we have

$$\begin{aligned} d_D \llbracket a + \alpha, b + \beta \rrbracket &= d_D(a \triangleright \beta + b \triangleright \alpha) \\ &= -\varphi_1(a \triangleright \beta) - \varphi_1(b \triangleright \alpha) + \tilde{d}(a \triangleright \beta) + \tilde{d}(b \triangleright \alpha), \end{aligned}$$

and

$$\begin{aligned} &\llbracket d_D(a + \alpha), b + \beta \rrbracket - \llbracket a + \alpha, d_D(b + \beta) \rrbracket \\ &= [da + \varphi_0 \alpha, b] - \sigma(da + \varphi_0 \alpha, \beta) + (da + \varphi_0 \alpha) \triangleright \beta \\ &\quad + [db + \varphi_0 \beta, a] - \sigma(db + \varphi_0 \beta, \alpha) + (db + \varphi_0 \beta) \triangleright \alpha \\ &= [\varphi_0 \alpha, b] + \sigma(\alpha, db) - [a, \varphi_0 \beta] - \sigma(da, \beta) + da \triangleright \beta + db \triangleright \alpha, \end{aligned}$$

where we have used conditions (i) and (iv) of Definition 3.2. Therefore, the equality $d_D \llbracket a + \alpha, b + \beta \rrbracket = \llbracket d_D(a + \alpha), b + \beta \rrbracket - \llbracket a + \alpha, d_D(b + \beta) \rrbracket$ holds since Π is a homomorphism.

- $n = 3$: We are supposed to check the graded Jacobi identity: for any $x, y, z \in \mathfrak{V}$,

$$\begin{aligned} & (-1)^{|x| \cdot |z|} \llbracket \llbracket x, y \rrbracket, z \rrbracket + c.p. \\ = & (-1)^{|x| \cdot |z| + 1} \{ d_D l^3(x, y, z) + l^3(d_D x, y, z) + (-1)^{|x|} l^3(x, d_D y, z) + (-1)^{|x| + |y|} l^3(x, y, d_D z) \}. \end{aligned}$$

Following from that \triangleright is an action and Π is a homomorphism, we have

$$\begin{aligned} \llbracket \llbracket x, y \rrbracket, z \rrbracket + c.p. &= -dl_3(x, y, z) = -d_D l^3(x, y, z), \\ \llbracket \llbracket x, y \rrbracket, \xi \rrbracket + c.p. &= (x, y) \triangleright \tilde{d}\xi = -l^3(x, y, d_D \xi), \end{aligned}$$

and

$$\begin{aligned} & \llbracket \llbracket x, y \rrbracket, a + \alpha \rrbracket + c.p. \\ = & \llbracket [x, y], a \rrbracket - \sigma([x, y], \alpha) + [x, y] \triangleright \alpha + \llbracket [y, a], x \rrbracket - [\sigma(y, \alpha), x] - \sigma(y \triangleright \alpha, x) - x \triangleright (y \triangleright \alpha) \\ & - \llbracket [x, a], y \rrbracket + [\sigma(x, \alpha), y] + \sigma(x \triangleright \alpha, y) + y \triangleright (x \triangleright \alpha) \\ = & -l_3(x, y, da) + \tilde{d}((x, y) \triangleright \alpha) - \varphi_1((x, y) \triangleright \alpha) - l_3(x, y, \varphi_0 \alpha) \\ = & -l^3(x, y, d_D(a + \alpha)) - d_D l^3(x, y, a + \alpha). \end{aligned}$$

The next case is,

$$\begin{aligned} & \llbracket \llbracket x, a + \alpha \rrbracket, b + \beta \rrbracket + \llbracket \llbracket a + \alpha, b + \beta \rrbracket, x \rrbracket - \llbracket \llbracket b + \beta, x \rrbracket, a + \alpha \rrbracket \\ = & [x, a] \triangleright \beta - \sigma(x, \alpha) \triangleright \beta + b \triangleright (x \triangleright \alpha) - x \triangleright (a \triangleright \beta) - x \triangleright (b \triangleright \alpha) \\ & + [x, b] \triangleright \alpha - \sigma(x, \beta) \triangleright \alpha + a \triangleright (x \triangleright \beta) \\ = & (x, da) \triangleright \beta + (x, db) \triangleright \alpha + (x, \varphi_0 \alpha) \triangleright \beta + (x, \varphi_0 \beta) \triangleright \alpha \\ = & -l^3(x, d_D(a + \alpha), b + \beta) + l^3(x, a + \alpha, d_D(b + \beta)), \end{aligned}$$

where we have used the equation $\sigma(x, \alpha) \triangleright \beta + \sigma(x, \beta) \triangleright \alpha = -(x, \varphi_0 \alpha) \triangleright \beta - (x, \varphi_0 \beta) \triangleright \alpha$ followed from condition (iii) of Definition 3.2. This finishes the proof of the graded Jacobi identity.

• $n = 4$: Specifically, for four elements in \mathfrak{V}_0 , (1) holds since \mathfrak{g} is a Lie 2-algebra and the definition of l^3 . While for three elements in \mathfrak{V}_0 and one element in \mathfrak{V}_1 , by straightforward deduce, condition (1) is equivalent to the coherence law of L_2 and L_3 on three elements in \mathfrak{g}_0 and one element in \mathfrak{m}_0 in Lemma 3.1. By careful analysis, all the other cases are trivial.

- $n = 5$: We shall prove

$$\Sigma_{\sigma} (-1)^{\sigma} K \operatorname{sgn}(\sigma) l^3(l^3(x_{\sigma_1}, x_{\sigma_2}, x_{\sigma_3}), x_{\sigma_4}, x_{\sigma_5}) = 0.$$

Actually, every term in the summation vanishes by the definition of l^3 . This completes the proof. ■

3.3 Examples

Example 3.6. Let $(\mathfrak{m}, \mathfrak{g}, \phi, \varphi)$ be a strong crossed module with strict Lie 2-algebras $\mathfrak{m}, \mathfrak{g}$ and strong homomorphism ϕ . Treating \mathfrak{g} as a crossed module of Lie algebras with $[a, b] = [da, b]$ on \mathfrak{g}_1 and \mathfrak{m} likewise, we get a commutative diagram

$$\begin{array}{ccc} \mathfrak{m}_1 & \xrightarrow{\varphi_1} & \mathfrak{g}_1 \\ \downarrow \tilde{d} & \searrow \varphi_0 \circ \tilde{d} & \downarrow d \\ \mathfrak{m}_0 & \xrightarrow{\varphi_0} & \mathfrak{g}_0 \end{array}$$

such that all the maps are crossed modules of Lie algebras. Moreover, defining $[a, \xi] = da \triangleright \xi, \forall a \in \mathfrak{g}_1, \xi \in \mathfrak{m}_1$, the crossed product $\mathfrak{g} \triangleright_{\phi} \mathfrak{m}$ is also a crossed module of Lie algebras.

Remark 3.7. *This example remains us of the notion of crossed squares and 2-crossed modules of Lie algebras introduced by Ellis in [8]. The group-theoretic setting is due to Conduché ([7]). See also [20, 9] for more details. The relation between the strong crossed modules in the above example and crossed squares of Lie algebras is still a mystery to us, which deserves to be further studied.*

Example 3.8. For a Lie 2-algebra $(\mathfrak{g}, d, l_2, l_3)$, it is obvious that $\text{Der}(\mathfrak{g})$ acts on \mathfrak{g} by derivations with $Id : \text{Der}(\mathfrak{g}) \rightarrow \text{Der}(\mathfrak{g})$. Consider the adjoint homomorphism $\overline{ad} : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$ given by (13) and a linear map $\sigma : \text{Der}_0(\mathfrak{g}) \wedge \mathfrak{g}_0 \rightarrow \text{Der}_1(\mathfrak{g})$ defined by

$$\sigma(X + l_X, x) = -l_X(x, \cdot), \quad \forall X + l_X \in \text{Der}_0(\mathfrak{g}), x \in \mathfrak{g}_0.$$

By straightforward verification, we get $(\mathfrak{g}, \text{Der}(\mathfrak{g}), Id, \overline{ad}, \sigma)$ is a crossed module. Note that this crossed module is not strong even if \mathfrak{g} is a strict Lie 2-algebra. From Theorem 3.5 it follows that the 3-term complex of vector spaces

$$\text{DER}(\mathfrak{g}) : \mathfrak{g}_1 \xrightarrow{d_D} \text{Der}_1(\mathfrak{g}) \oplus \mathfrak{g}_0 \xrightarrow{d_D} \text{Der}_0(\mathfrak{g})$$

is a strict Lie 3-algebra. Moreover, l^3 vanishes by definition. This recovers [5, Theorem 3.8].

Example 3.9. Let $(\mathfrak{g}, d, l_2, l_3)$ be a Lie 2-algebra and \mathfrak{m} an ideal of \mathfrak{g} , then $(\mathfrak{m}, \mathfrak{g}, \overline{ad}, i)$ is a strong crossed module, in which $\overline{ad} : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{m})$ given by (13) is an action of \mathfrak{g} on \mathfrak{m} by derivations and the inclusion map $i : \mathfrak{m} \rightarrow \mathfrak{g}$ is a strong homomorphism.

According to Example 3.8 and Example 3.9 with $\mathfrak{m} = \mathfrak{g}$, there are two natural crossed modules for a Lie 2-algebra \mathfrak{g} . Moreover, we get a commutative diagram

$$\begin{array}{ccc} \mathfrak{g} \triangleright_{\overline{ad}} \mathfrak{g} & \xrightarrow{\overline{ad} \oplus Id} & \text{Der}(\mathfrak{g}) \triangleright_{Id} \mathfrak{g} \\ Id + Id \downarrow & & \downarrow Id + \sigma + \overline{ad} \\ \mathfrak{g} & \xrightarrow{\overline{ad}} & \text{Der}(\mathfrak{g}) \end{array}$$

such that all the maps are Lie 2-algebra homomorphisms. Namely, $(Id, \overline{ad}, 0)$ is a homomorphism between the two crossed modules.

Example 3.10. The following example is inspired by [24, Example 3]. Given a Lie 2-algebra \mathfrak{h} , a short exact sequence of \mathfrak{h} -modules:

$$0 \longrightarrow \mathbb{V} \xrightarrow{p} \mathbb{I} \xrightarrow{q} \mathbb{Q} \longrightarrow 0, \quad (15)$$

(regarded as a short exact sequence of trivial Lie 2-algebras) and a 2-cocycle $\lambda \in C^2(\mathfrak{h}, \mathbb{Q})$, by [15, Theorem 4.5], we get an abelian extension of \mathfrak{h} by \mathbb{Q} :

$$0 \rightarrow \mathbb{Q} \xrightarrow{i} \mathfrak{h} \oplus_{\lambda} \mathbb{Q} \xrightarrow{\pi} \mathfrak{h} \rightarrow 0. \quad (16)$$

Splicing (15) and (16) together, we get a strong crossed module $\varepsilon = (\mathbb{I}, \mathfrak{h} \oplus_{\lambda} \mathbb{Q}, \phi, \varphi)$, where $\mathbb{I} \xrightarrow{\varphi} \mathfrak{h} \oplus_{\lambda} \mathbb{Q}$ is defined by $\varphi(v) = (0, q(v))$, and ϕ is an action of $\mathfrak{h} \oplus_{\lambda} \mathbb{Q}$ on \mathbb{I} given by the action of \mathfrak{h} on \mathbb{I} . Since \mathbb{I} is a trivial Lie 2-algebra, by defining $l_{\phi_0} = 0$, it is clear that ϕ is an action by derivations.

4 The first cohomology and derivations of crossed modules

4.1 Lie algebra structures on $C^1(\mathfrak{g}, \mathbb{V})$

Let $\mathbb{V} : V_1 \xrightarrow{\partial} V_0$ and $\mathbb{W} : W_1 \xrightarrow{d} W_0$ be two 2-vector spaces. Then we can construct a new 2-vector space

$$\text{Hom}(\mathbb{V}, \mathbb{W}) : \text{Hom}_1(\mathbb{V}, \mathbb{W}) \xrightarrow{\delta} \text{Hom}_0(\mathbb{V}, \mathbb{W})$$

where $\text{Hom}_1(\mathbb{V}, \mathbb{W}) = \text{Hom}(V_0, W_1)$,

$$\text{Hom}_0(\mathbb{V}, \mathbb{W}) = \{X_0 + X_1 \in \text{Hom}(V_0, W_0) \oplus \text{Hom}(V_1, W_1) | X_0 \circ \partial = d \circ X_1\},$$

and $\delta(\Theta) = d \circ \Theta + \Theta \circ \partial$.

Lemma 4.1. *Given $\varphi \in \text{Hom}_0(\mathbb{W}, \mathbb{V})$, there exists a strict Lie 2-algebra structure on $\text{Hom}(\mathbb{V}, \mathbb{W})$, where the bracket $[\cdot, \cdot]_\varphi$ is given by:*

$$[X, Y]_\varphi = X \circ \varphi \circ Y - Y \circ \varphi \circ X, \quad [X, \Theta]_\varphi = X \circ \varphi \circ \Theta - \Theta \circ \varphi \circ X, \quad [\Theta, \Theta']_\varphi = 0. \quad (17)$$

In particular, if $\mathbb{W} = \mathbb{V}$, $\varphi = Id$, this recovers the strict Lie 2-algebra $\text{End}(\mathbb{V})$, which plays the same role as $\text{gl}(V)$ for a vector space V .

Set $L(\mathfrak{g}, \mathbb{V}) \triangleq \text{Hom}(\mathfrak{g}_0, V_0) \oplus \text{Hom}(\mathfrak{g}_1, V_1)$, then $C^1(\mathfrak{g}, \mathbb{V}) = L(\mathfrak{g}, \mathbb{V}) \oplus \text{Hom}(\wedge^2 \mathfrak{g}_0, V_1)$. Given an element $\varphi = (\varphi_0, \varphi_1) \in \text{Hom}(V_0, \mathfrak{g}_0) \oplus \text{Hom}(V_1, \mathfrak{g}_1)$, we have $L(\mathfrak{g}, \mathbb{V})$ is a Lie algebra with the bracket $[\cdot, \cdot]_\varphi$ given by the first formula of (17). Furthermore, we have:

Lemma 4.2. (1) $\text{Hom}(\wedge^2 \mathfrak{g}_0, V_1)$ is an $L(\mathfrak{g}, \mathbb{V})$ -module, where the action is given by

$$(X \triangleright \xi)(x, y) = X_1 \varphi_1 \xi(x, y) - \xi(\varphi_0 X_0 x, y) - \xi(x, \varphi_0 X_0 y), \quad \forall x, y \in \mathfrak{g}_0,$$

for any $X = (X_0, X_1) \in L(\mathfrak{g}, \mathbb{V})$ and $\xi \in \text{Hom}(\wedge^2 \mathfrak{g}_0, V_1)$.

(2) let $\sigma : \mathfrak{g}_0 \wedge V_0 \longrightarrow \mathfrak{g}_1$ be a linear map satisfying $\sigma(\varphi_0 u, v) = \sigma(u, \varphi_0 v), \forall u, v \in V_0$. Then ω^σ defined by

$$\omega^\sigma(X, Y)(x, y) \triangleq X \sigma(Yx, y) + X \sigma(x, Yy) - Y \sigma(Xx, y) - Y \sigma(x, Xy)$$

is a 2-cocycle of $L(\mathfrak{g}, \mathbb{V})$ with values in $\text{Hom}(\wedge^2 \mathfrak{g}_0, V_1)$.

(3) $C^1(\mathfrak{g}, \mathbb{V})$ is a Lie algebra with the bracket $\{\cdot, \cdot\}$ given by

$$\{X + \xi, Y + \eta\} = [X, Y]_\varphi + X \triangleright \eta - Y \triangleright \xi + \omega^\sigma(X, Y).$$

Proof. To prove that \triangleright is an action, we need to check

$$[Y, X]_\varphi \triangleright \xi = Y \triangleright (X \triangleright \xi) - X \triangleright (Y \triangleright \xi).$$

It follows from

$$\begin{aligned} & (Y \triangleright (X \triangleright \xi) - X \triangleright (Y \triangleright \xi))(x, y) \\ &= Y \varphi(X \triangleright \xi)(x, y) - X \triangleright \xi(\varphi Y x, y) - X \triangleright \xi(x, \varphi Y y) - c.p.(X, Y) \\ &= Y \varphi(X \varphi \xi(x, y) - \xi(\varphi X x, y) - \xi(x, \varphi X y)) - X \varphi \xi(\varphi Y x, y) + \xi(\varphi X \varphi Y x, y) + \xi(\varphi Y x, \varphi X y) \\ & \quad - X \varphi \xi(x, \varphi Y y) + \xi(\varphi X x, \varphi Y y) + \xi(x, \varphi X \varphi Y y) - c.p.(X, Y) \\ &= [Y, X]_\varphi \varphi \xi(x, y) - \xi(\varphi[Y, X]_\varphi x, y) - \xi(x, \varphi[Y, X]_\varphi y) \\ &= ([Y, X]_\varphi \triangleright \xi)(x, y). \end{aligned}$$

It is direct to verify that ω^σ is a 2-cocycle. We omit the details. ■

4.2 Derivations of crossed modules

The main goal of this section is to propose a notion of derivations of a crossed module of Lie 2-algebras and prove that it is a strict Lie 2-algebra, which generalises the derivations of a Lie 2-algebra $\text{Der}(\mathfrak{g})$.

Given a crossed module $(\mathfrak{m}, \mathfrak{g}, \phi, \varphi, \sigma)$, since \mathfrak{m} is a \mathfrak{g} -module, we have a natural complex

$$\text{Der}(\mathfrak{g}, \mathfrak{m}) : \text{Der}_1(\mathfrak{g}, \mathfrak{m}) \triangleq \text{Hom}(\mathfrak{g}_0, \mathfrak{m}_1) \xrightarrow{-D} \text{Der}_0(\mathfrak{g}, \mathfrak{m}),$$

where $\text{Der}_0(\mathfrak{g}, \mathfrak{m})$ is the set of 1-cocycles and D is the Lie 2-algebra coboundary operator. Denote by $\text{Inn}_0(\mathfrak{g}, \mathfrak{m})$ the set of 1-coboundaries. Explicitly, by (8) and (9), we have $-D(\Theta) = \delta(\Theta) + l_{\delta(\Theta)}$ and

$$l_{\delta(\Theta)}(x, y) = \Theta[x, y] - x \triangleright \Theta y + y \triangleright \Theta x, \quad \forall x, y \in \mathfrak{g}_0, \Theta \in \text{Der}_1(\mathfrak{g}, \mathfrak{m}).$$

We call $\text{Der}(\mathfrak{g}, \mathfrak{m})$ the **derivations of the crossed module**.

By condition (iv) of Definition 3.2 and Lemma 4.2, $(C^1(\mathfrak{g}, \mathfrak{m}), \{\cdot, \cdot\})$ is a Lie algebra. As its subspace, $\text{Der}_0(\mathfrak{g}, \mathfrak{m})$ inherits a bracket operation, that is, for any $X + l_X, Y + l_Y \in \text{Der}_0(\mathfrak{g}, \mathfrak{m})$,

$$\{X + l_X, Y + l_Y\} = [X, Y]_{\varphi} + X \triangleright l_Y - Y \triangleright l_X + \omega^{\sigma}(X, Y).$$

Set $l_{[X, Y]_{\varphi}} = X \triangleright l_Y - Y \triangleright l_X + \omega^{\sigma}(X, Y)$. To be more precise,

$$l_{[X, Y]_{\varphi}}(x, y) = X\varphi l_Y(x, y) - l_Y(\varphi X x, y) - l_Y(x, \varphi X y) + X\sigma(Yx, y) + X\sigma(x, Yy) - c.p.(X, Y).$$

Define $\{X + l_X, \Theta\} \triangleq [X, \Theta]_{\varphi}$.

Lemma 4.3. *With the above notations, $\text{Der}_0(\mathfrak{g}, \mathfrak{m})$ is a Lie subalgebra of $(C^1(\mathfrak{g}, \mathfrak{m}), \{\cdot, \cdot\})$.*

Proof. For any $X + l_X, Y + l_Y \in \text{Der}_0(\mathfrak{g}, \mathfrak{m})$, we need to prove that $[X, Y]_{\varphi} + l_{[X, Y]_{\varphi}}$ is a 1-cocycle. It is not hard to check conditions (4)-(6). We just verify condition (7), i.e., for any $x, y, z \in \mathfrak{g}_0$,

$$l_{[X, Y]_{\varphi}}(x, [y, z]) + x \triangleright l_{[X, Y]_{\varphi}}(y, z) - (y, z) \triangleright [X, Y]_{\varphi} x + c.p.(x, y, z) = [X, Y]_{\varphi} l_3(x, y, z). \quad (18)$$

Firstly, as $X + l_X, Y + l_Y \in \text{Der}_0(\mathfrak{g}, \mathfrak{m})$ and $\Pi = Id + \sigma + \varphi$ is a homomorphism, we have

$$\begin{aligned} \varphi X[y, z] &= \varphi(y \triangleright Xz - z \triangleright Xy + \tilde{d}l_X(y, z)) \\ &= [y, \varphi Xz] + d\sigma(y, Xz) - [z, \varphi Xy] - d\sigma(z, Xy) + \varphi \tilde{d}l_X(y, z), \end{aligned} \quad (19)$$

$$\begin{aligned} X\varphi(Yl_3(x, y, z)) &= X\varphi(l_Y(x, [y, z]) + x \triangleright l_Y(y, z) - (y, z) \triangleright Yx + c.p.(x, y, z)) \\ &= X\varphi l_Y(x, [y, z]) + x \triangleright X\varphi l_Y(y, z) - \varphi l_Y(y, z) \triangleright Xx + l_X(x, d\varphi l_Y(y, z)) \\ &\quad + X\sigma(x, \tilde{d}l_Y(y, z)) - X\varphi((y, z) \triangleright Yx) + c.p.(x, y, z), \end{aligned} \quad (20)$$

and

$$\begin{aligned} &-X(\varphi((y, z) \triangleright Yx) + l_3(\varphi Yx, y, z)) \\ &= X(\sigma(Yx, [y, z]) + \sigma(y, z \triangleright Yx) - \sigma(z, y \triangleright Yx) + [y, \sigma(z, Yx)] + [z, \sigma(Yx, y)]) \\ &= X\sigma(Yx, [y, z]) + X\sigma(y, z \triangleright Yx) - X\sigma(z, y \triangleright Yx) + y \triangleright X\sigma(z, Yx) - \sigma(z, Yx) \triangleright Xy \\ &\quad + l_X(y, d\sigma(z, Yx)) - z \triangleright X\sigma(y, Yx) + \sigma(y, Yx) \triangleright Xz - l_X(z, d\sigma(y, Yx)). \end{aligned} \quad (21)$$

Thus, (18) follows from

$$\begin{aligned}
& l_{[X,Y]_\varphi}(x, [y, z]) + x \triangleright l_{[X,Y]_\varphi}(y, z) - (y, z) \triangleright [X, Y]_\varphi x + c.p.(x, y, z) \\
= & X\varphi l_Y(x, [y, z]) - l_Y(x, [y, \varphi Xz] + d\sigma(y, Xz) - [z, \varphi Xy] - d\sigma(z, Xy) + \varphi \tilde{d}l_X(y, z)) \\
& - l_Y(\varphi Xx, [y, z]) + X\sigma(Yx, [y, z]) + X\sigma(x, y \triangleright Yz - z \triangleright Yy + \tilde{d}l_Y(y, z)) \\
& + x \triangleright (X\varphi l_Y(y, z) - l_Y(\varphi Xy, z) - l_Y(y, \varphi Xz) + X\sigma(Yy, z) + X\sigma(y, Yz)) \\
& - (y, z) \triangleright X\varphi Yx - c.p.(X, Y) + c.p.(x, y, z) \quad \text{by (19)} \\
= & (X\varphi l_Y(x, [y, z]) + x \triangleright X\varphi l_Y(y, z) + l_X(x, d\varphi l_Y(y, z)) + X\sigma(x, \tilde{d}l_Y(y, z))) \\
& - (l_Y(\varphi Xx, [y, z]) + c.p.(\varphi Xx, y, z) + y \triangleright l_Y(z, \varphi Xx) + z \triangleright l_Y(\varphi Xx, y) - (y, z) \triangleright Y\varphi Xx) \\
& + (X\sigma(Yx, [y, z]) + c.p.(Yx, y, z) + y \triangleright X\sigma(z, Yx) + l_X(y, d\sigma(z, Yx)) \\
& - z \triangleright X\sigma(y, Yx) - l_X(z, d\sigma(y, Yx))) - c.p.(X, Y) + c.p.(x, y, z) \\
= & X\varphi Yl_3(x, y, z) + (\varphi l_Y(y, z) \triangleright Xx + X\varphi((y, z) \triangleright Yx) \\
& - Yl_3(\varphi Xx, y, z) + \varphi Xx \triangleright l_Y(y, z) - (z, \varphi Xx) \triangleright Yy - (\varphi Xx, y) \triangleright Yz \\
& - X\varphi((y, z) \triangleright Yx) - Xl_3(\varphi Yx, y, z) + \sigma(z, Yx) \triangleright Xy - \sigma(y, Yx) \triangleright Xz + c.p.(x, y, z)) \\
& - c.p.(X, Y) \quad \text{by (20), (7) and (21)} \\
= & l_{\phi_0(z)}(Xx, Yy) - l_{\phi_0(y)}(Xx, Yz) + c.p.(x, y, z) - c.p.(X, Y) + [X, Y]_\varphi l_3(x, y, z) \\
= & [X, Y]_\varphi l_3(x, y, z),
\end{aligned}$$

where the penultimate equation holds since conditions (i) and (iii) of Definition 3.2. ■

Theorem 4.4. *For a crossed module $(\mathfrak{m}, \mathfrak{g}, \phi, \varphi, \sigma)$, its derivations $(\text{Der}(\mathfrak{g}, \mathfrak{m}), \{\cdot, \cdot\})$ is a strict Lie 2-algebra.*

Proof. By Lemma 4.1 and Lemma 4.3, we only need to verify that $-D$ is a graded derivation with respect to the bracket operation $\{\cdot, \cdot\}$, i.e., for any $X + l_X \in \text{Der}_0(\mathfrak{g}, \mathfrak{m})$, $\Theta, \Theta' \in \text{Der}_1(\mathfrak{g}, \mathfrak{m})$,

$$-D\{X + l_X, \Theta\} = \{X + l_X, -D\Theta\}, \quad (22)$$

$$\{-D\Theta, \Theta'\} = \{\Theta, -D\Theta'\}. \quad (23)$$

The left hand side of (22) is equal to $\delta([X, \Theta]_\varphi) + l_{\delta([X, \Theta]_\varphi)}$. By repeatedly applying condition (i)

of Definition 3.2 and the fact that Π is a homomorphism, we have

$$\begin{aligned}
& l_{\delta([X, \Theta]_\varphi)}(x, y) \\
= & [X, \Theta]_\varphi[x, y] - x \triangleright [X, \Theta]_\varphi y + y \triangleright [X, \Theta]_\varphi x \\
= & X\varphi(x \triangleright \Theta y - y \triangleright \Theta x + l_{\delta(\Theta)}(x, y)) - \Theta\varphi(x \triangleright Xy - y \triangleright Xx + \tilde{d}l_X(x, y)) \\
& - x \triangleright (X\varphi\Theta y - \Theta\varphi Xy) + y \triangleright (X\varphi\Theta x - \Theta\varphi Xx) \\
= & X([x, \varphi\Theta y] + \sigma(x, \tilde{d}\Theta y) - [y, \varphi\Theta x] - \sigma(y, \tilde{d}\Theta x)) + \varphi l_{\delta(\Theta)}(x, y) \\
& - \Theta([x, \varphi Xy] + d\sigma(x, Xy) - [y, \varphi Xx] - d\sigma(y, Xx) + \varphi \tilde{d}l_X(x, y)) \\
& - x \triangleright (X\varphi\Theta y - \Theta\varphi Xy) + y \triangleright (X\varphi\Theta x - \Theta\varphi Xx) \\
= & x \triangleright X\varphi\Theta y - \varphi\Theta y \triangleright Xx + l_X(x, d\varphi\Theta y) + X\sigma(x, \tilde{d}\Theta y) \\
& - y \triangleright X\varphi\Theta x + \varphi\Theta x \triangleright Xy - l_X(y, d\varphi\Theta x) - X\sigma(y, \tilde{d}\Theta x) + X\varphi l_{\delta(\Theta)}(x, y) \\
& - x \triangleright \Theta\varphi Xy + \varphi Xy \triangleright \Theta x - l_{\delta(\Theta)}(x, \varphi Xy) - \Theta d\sigma(x, Xy) \\
& + y \triangleright \Theta\varphi Xx - \varphi Xx \triangleright \Theta y + l_{\delta(\Theta)}(y, \varphi Xx) + \Theta d\sigma(y, Xx) - \Theta\varphi \tilde{d}l_X(x, y) \\
& - x \triangleright (X\varphi\Theta y - \Theta\varphi Xy) + y \triangleright (X\varphi\Theta x - \Theta\varphi Xx) \\
= & l_X(x, d\varphi\Theta y) + X\sigma(x, \tilde{d}\Theta y) - l_X(y, d\varphi\Theta x) - X\sigma(y, \tilde{d}\Theta x) + X\varphi l_{\delta(\Theta)}(x, y) \\
& - l_{\delta(\Theta)}(x, \varphi Xy) - \Theta d\sigma(x, Xy) + l_{\delta(\Theta)}(y, \varphi Xx) + \Theta d\sigma(y, Xx) - \Theta\varphi \tilde{d}l_X(x, y) \\
= & l_{[X, \delta(\Theta)]_\varphi}(x, y).
\end{aligned}$$

The right hand side of (22) is equal to

$$\{X + l_X, \delta(\Theta) + l_{\delta(\Theta)}\} = [X, \delta(\Theta)]_\varphi + l_{[X, \delta(\Theta)]_\varphi}.$$

Thus, (22) holds since $\delta([X, \Theta]_\varphi) = [X, \delta(\Theta)]_\varphi$.

The equation (23) is a consequence of $[\delta(\Theta), \Theta']_\varphi = [\Theta, \delta(\Theta')]_\varphi$. This finishes the proof. ■

Proposition 4.5. *For a crossed module $(\mathfrak{m}, \mathfrak{g}, \phi, \varphi, \sigma)$, the 1st cohomology group $H^1(\mathfrak{g}, \mathfrak{m}) = \text{Der}_0(\mathfrak{g}, \mathfrak{m})/\text{Inn}_0(\mathfrak{g}, \mathfrak{m})$ is a quotient Lie algebra.*

Proof. We shall prove $\text{Inn}_0(\mathfrak{g}, \mathfrak{m})$ is an ideal of $\text{Der}_0(\mathfrak{g}, \mathfrak{m})$. Since (22) holds, it suffices to show

$$\{X + l_X, -D\alpha\} = -D(X\varphi\alpha + l_X(\varphi\alpha, \cdot) + X\sigma(\alpha, \cdot)), \quad \forall \alpha \in \mathfrak{m}_0. \quad (24)$$

Acting on $x \in \mathfrak{g}_0$ and taking into account (8) and (9), we have

$$\begin{aligned}
\{X + l_X, -D\alpha\}(x) &= -X\varphi(D\alpha)x + (D\alpha)\varphi Xx \\
&= -X\varphi(x \triangleright \alpha) + \varphi Xx \triangleright \alpha \\
&= -x \triangleright X\varphi\alpha + \varphi\alpha \triangleright Xx - \tilde{d}l_X(x, \varphi\alpha) - Xd\sigma(x, \alpha) + \varphi Xx \triangleright \alpha \\
&= -x \triangleright X\varphi\alpha - \tilde{d}l_X(x, \varphi\alpha) - \tilde{d}X\sigma(x, \alpha) \\
&= -D(X\varphi\alpha + l_X(\varphi\alpha, \cdot) + X\sigma(\alpha, \cdot))(x),
\end{aligned}$$

where we have used condition (i) and Π is a homomorphism in Definition 3.2. Likewise, (24) holds on \mathfrak{g}_1 . Finally, relying on the coherence law of the homomorphism Π , we have

$$\begin{aligned}
& X(\sigma([x, y], \alpha) + \sigma(y \triangleright \alpha, x) - \sigma(x \triangleright \alpha, y) - \varphi((x, y) \triangleright \alpha) - l_3(x, y, \varphi\alpha)) \\
= & X([x, \sigma(y, \alpha)] + [y, \sigma(x, \alpha)]) \\
= & x \triangleright X\sigma(y, \alpha) - \sigma(y, \alpha) \triangleright Xx + l_X(x, d\sigma(y, \alpha)) \\
& + y \triangleright X\sigma(x, \alpha) - \sigma(x, \alpha) \triangleright Xy + l_X(y, d\sigma(x, \alpha)).
\end{aligned} \quad (25)$$

Thus, acting on $\wedge^2 \mathfrak{g}_0$, it turns out that

$$\begin{aligned}
& \{X + l_X, -D\alpha\}(x, y) \\
&= -X\varphi D\alpha(x, y) + D\alpha(\varphi Xx, y) + D\alpha(x, \varphi Xy) + D\alpha(\varphi l_X(x, y)) - l_X(\varphi D\alpha(x, y)) \\
&\quad - l_X(x, \varphi D\alpha(y)) - X\sigma(D\alpha(x), y) - X\sigma(x, D\alpha(y)) + D\alpha(\sigma(Xx, y)) + D\alpha(\sigma(x, Xy)) \\
&= -X\varphi((x, y) \triangleright \alpha) + (\varphi Xx, y) \triangleright \alpha + (x, \varphi Xy) \triangleright \alpha + \varphi l_X(x, y) \triangleright \alpha - l_X([x, \varphi \alpha] + d\sigma(x, \alpha), y) \\
&\quad - l_X(x, [y, \varphi \alpha] + d\sigma(y, \alpha)) - X\sigma(x \triangleright \alpha, y) - X\sigma(x, y \triangleright \alpha) + \sigma(Xx, y) \triangleright \alpha + \sigma(x, Xy) \triangleright \alpha \\
&= (x \triangleright X\sigma(y, \alpha) - \sigma(y, \alpha) \triangleright Xx + y \triangleright X\sigma(\alpha, x) - \sigma(\alpha, x) \triangleright Xy - X\sigma([x, y], \alpha) + Xl_3(x, y, \varphi \alpha)) \\
&\quad + (l_X(\varphi \alpha, [x, y]) - (x, y) \triangleright X\varphi \alpha - (y, \varphi \alpha) \triangleright Xx - (\varphi \alpha, x) \triangleright Xy + x \triangleright l_X(y, \varphi \alpha) \\
&\quad + y \triangleright l_X(\varphi \alpha, x) - Xl_3(x, y, \varphi \alpha)) + l_{\phi_0(y)}(Xx, \alpha) - l_{\phi_0(x)}(Xy, \alpha) \quad \text{by (25) and (7)} \\
&= x \triangleright X\sigma(y, \alpha) + y \triangleright X\sigma(\alpha, x) - X\sigma([x, y], \alpha) + l_X(\varphi \alpha, [x, y]) + x \triangleright l_X(y, \varphi \alpha) + y \triangleright l_X(\varphi \alpha, x) \\
&\quad - (x, y) \triangleright X\varphi \alpha + (l_{\phi_0(y)}(\alpha, Xx) - l_{\phi_0(x)}(\alpha, Xy) + l_{\phi_0(y)}(Xx, \alpha) - l_{\phi_0(x)}(Xy, \alpha)) \\
&= -D(X\varphi \alpha + l_X(\varphi \alpha, \cdot) + X\sigma(\alpha, \cdot))(x, y),
\end{aligned}$$

where we have used (iii) of Definition 3.2. The proof is finished. ■

Remark 4.6. Let $\mathfrak{m} = \mathfrak{g}, i = Id$ in Example 3.9. Then Theorem 4.4 recovers the theorem that $\text{Der}(\mathfrak{g})$ is a strict Lie 2-algebra in [5] and Proposition 4.5 implies that $H^1(\mathfrak{g}) = \text{Der}_0(\mathfrak{g})/\text{Inn}_0(\mathfrak{g})$ is a quotient Lie algebra. Also, it justifies our definition of $\text{Inn}_0(\mathfrak{g})$, while it is defined by $\text{Im}(D|_{\mathfrak{g}_0})$ in [5] which is not an ideal of $\text{Der}_0(\mathfrak{g})$ by (24).

Remark 4.7. It should be interesting to consider the strict 2-group integrated from the strict Lie 2-algebra $\text{Der}(\mathfrak{g}, \mathfrak{m})$. In particular, for $\text{Der}(\mathfrak{g})$, we can explicate the strict 2-group $\text{Aut}(\mathfrak{g})$ consisted of all the automorphisms of Lie 2-algebra \mathfrak{g} . Furthermore, for a strict Lie 2-algebra \mathfrak{g} with corresponding strict Lie 2-groups \mathcal{G} , it is interesting to establish the connection between $\text{Der}(\mathfrak{g})$ and the automorphism 2-group $\text{Aut}(\mathcal{G})$ ([21]).

5 Classification of strong crossed modules via H^3

5.1 Crossed modules from extensions of short exact sequences

Suppose $(\mathfrak{g}, d, [\cdot, \cdot], l_3)$ is a Lie 2-algebra and $\mathfrak{k} \subset \mathfrak{g}$ is an ideal. This gives rise to a short exact sequence of Lie 2-algebras

$$0 \rightarrow \mathfrak{k} \hookrightarrow \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \rightarrow 0,$$

where $\mathfrak{h} = \mathfrak{g}/\mathfrak{k}$ and π is the canonical projection, which is a strong Lie 2-algebra homomorphism. Furthermore, given an \mathfrak{h} -module \mathbb{V} , it is natural that \mathbb{V} endows with a \mathfrak{g} -module structure and then a trivial \mathfrak{h} -module structure as follows:

$$x \triangleright u = \tilde{x} \triangleright u, \quad (x, y) \triangleright u = (\tilde{x}, \tilde{y}) \triangleright u, \quad \forall x, y \in \mathfrak{g}, u \in \mathbb{V}, \tilde{x} = \pi(x), \tilde{y} = \pi(y).$$

The natural projection $\pi : \mathfrak{g} \rightarrow \mathfrak{h}$ induces a map of cochain complexes:

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{D^{\mathfrak{h}}} & C^2(\mathfrak{h}, \mathbb{V}) & \xrightarrow{D^{\mathfrak{h}}} & C^3(\mathfrak{h}, \mathbb{V}) & \xrightarrow{D^{\mathfrak{h}}} & \cdots \\
& & \pi^* \downarrow & & \pi^* \downarrow & & \\
\cdots & \xrightarrow{D^{\mathfrak{g}}} & C^2(\mathfrak{g}, \mathbb{V}) & \xrightarrow{D^{\mathfrak{g}}} & C^3(\mathfrak{g}, \mathbb{V}) & \xrightarrow{D^{\mathfrak{g}}} & \cdots
\end{array} \tag{26}$$

In this section, we always suppose $\alpha, \beta, \gamma \in \mathfrak{k}_0, \xi \in \mathfrak{k}_1, u, v, w \in V_0, m \in V_1$, and $x, y \in \mathfrak{g}_0, a \in \mathfrak{g}_1$.

Lemma 5.1. For a 2-cochain $\lambda \in C^2(\mathfrak{g}, \mathbb{V})$, the following three statements are equivalent:

- (1) $i_e(D^\mathfrak{g}\lambda) = 0, \forall e \in \mathfrak{k}$;
- (2) $D^\mathfrak{g}\lambda = \pi^*\theta$ for a 3-cocycle $\theta \in C^3(\mathfrak{h}, \mathbb{V})$;
- (3) The map $\phi^\lambda = (\phi_0^\lambda, \phi_1^\lambda, \phi_2^\lambda) : \mathfrak{g} \rightarrow \text{End}(\mathfrak{k} \oplus_\lambda \mathbb{V})$ given below defines an action of \mathfrak{g} on $\mathfrak{k} \oplus_\lambda \mathbb{V}$, where

$$\mathfrak{k} \oplus_\lambda \mathbb{V} : \mathfrak{k}_1 \oplus V_1 \xrightarrow{d^\lambda} \mathfrak{k}_0 \oplus V_0$$

is a 2-vector space with $d^\lambda(\xi + m) = d\xi + d^\mathbb{V}m + \lambda_0(\xi)$ and

$$\begin{cases} x \triangleright_\lambda (\alpha + u) &= [x, \alpha] + \lambda_1(x, \alpha) + \tilde{x} \triangleright u, \\ x \triangleright_\lambda (\xi + m) &= [x, \xi] + \lambda_2(x, \xi) + \tilde{x} \triangleright m, \\ a \triangleright_\lambda (\alpha + u) &= [a, \alpha] + \lambda_2(a, \alpha) + \tilde{a} \triangleright u, \\ (x, y) \triangleright_\lambda (\alpha + u) &= -l_3(x, y, \alpha) - \lambda_3(x, y, \alpha) + (\tilde{x}, \tilde{y}) \triangleright u. \end{cases} \quad (27)$$

That is, ϕ^λ is a Lie 2-algebra homomorphism.

Proof. It is obvious that (1) \Leftrightarrow (2), so it suffices to prove (1) \Leftrightarrow (3). Suppose that λ satisfies (1), we shall show that $\phi_0^\lambda(x) \in \text{End}_0(\mathfrak{k} \oplus_\lambda \mathbb{V})$ and ϕ^λ is a Lie 2-algebra homomorphism. Referring to (10), we use subscripts to distinguish the five components in $C^3(\mathfrak{g}, \mathbb{V})$.

Firstly, the equality $\phi_0^\lambda(x) \circ d^\lambda = d^\lambda \circ \phi_0^\lambda(x)$ holds since

$$\begin{aligned} & (\phi_0^\lambda(x) \circ d^\lambda - d^\lambda \circ \phi_0^\lambda(x))(\xi + m) \\ &= x \triangleright_\lambda (d\xi + d^\mathbb{V}m + \lambda_0(\xi)) - d^\lambda([x, \xi] + \lambda_2(x, \xi) + \tilde{x} \triangleright m) \\ &= [x, d\xi] + \lambda_1(x, d\xi) + \tilde{x} \triangleright (d^\mathbb{V}m + \lambda_0(\xi)) - d[x, \xi] - d^\mathbb{V}\lambda_2(x, \xi) - d^\mathbb{V}(\tilde{x} \triangleright m) - \lambda_0[x, \xi] \\ &= \lambda_1(x, d\xi) + \tilde{x} \triangleright \lambda_0(\xi) - d^\mathbb{V}\lambda_2(x, \xi) - \lambda_0[x, \xi] \\ &= (D^\mathfrak{g}\lambda)_0(x, \xi) \\ &= 0, \end{aligned}$$

where we have used the fact that \triangleright is an action. Analogously, we get

$$\begin{aligned} (\phi_0^\lambda(da) - \delta\phi_1^\lambda(a))(\alpha + u) &= -(D^\mathfrak{g}\lambda)_0(\alpha, a) = 0, \\ (\phi_0^\lambda(da) - \delta\phi_1^\lambda(a))(\xi + m) &= -(D^\mathfrak{g}\lambda)_1(a, \xi) = 0. \end{aligned}$$

Secondly, by an elementary computation, we obtain

$$\begin{aligned} & (\phi_0^\lambda[x, y] - [\phi_0^\lambda(x), \phi_0^\lambda(y)] - \delta\phi_2^\lambda(x, y))(\alpha + u) \\ &= [[x, y], \alpha] + \lambda_1([x, y], \alpha) + \widetilde{[x, y]} \triangleright u - [x, [y, \alpha]] - \lambda_1(x, [y, \alpha]) - \tilde{x} \triangleright (\lambda_1(y, \alpha) + \tilde{y} \triangleright u) \\ &\quad + [y, [x, \alpha]] + \lambda_1(y, [x, \alpha]) + \tilde{y} \triangleright (\lambda_1(x, \alpha) + \tilde{x} \triangleright u) + dl_3(x, y, \alpha) \\ &\quad + d^\mathbb{V}\lambda_3(x, y, \alpha) - d^\mathbb{V}((\tilde{x}, \tilde{y}) \triangleright u) + \lambda_0 l_3(x, y, \alpha) \\ &= \lambda_1([x, y], \alpha) + c.p. - \tilde{x} \triangleright \lambda_1(y, \alpha) + \tilde{y} \triangleright \lambda_1(x, \alpha) + d^\mathbb{V}\lambda_3(x, y, \alpha) + \lambda_0 l_3(x, y, \alpha) \\ &= -(D^\mathfrak{g}\lambda)_2(x, y, \alpha) \\ &= 0, \end{aligned}$$

where we have used the general Jacobi identity of \mathfrak{g} and the fact that \triangleright is an action. Likewise, we can deduce that

$$\begin{aligned} (\phi_0^\lambda[x, y] - [\phi_0^\lambda(x), \phi_0^\lambda(y)] - \delta\phi_2^\lambda(x, y))(\xi + m) &= -(D^\mathfrak{g}\lambda)_3(x, y, \xi) = 0, \\ (\phi_1^\lambda[x, a] - [\phi_0^\lambda(x), \phi_1^\lambda(a)] - \delta\phi_2^\lambda(x, da))(\alpha + u) &= -(D^\mathfrak{g}\lambda)_3(x, a, \alpha) = 0. \end{aligned}$$

Finally, for the coherence condition, following from the fact that \triangleright is an action and the coherence law of $[\cdot, \cdot]$ and l_3 , we have

$$\begin{aligned}
& ([\phi_0^\lambda(x), \phi_2^\lambda(y, z)] - \phi_2^\lambda([x, y], z) + c.p. - \phi_1^\lambda l_3(x, y, z))(\alpha + u) \\
= & -[x, l_3(y, z, \alpha)] - \lambda_2(x, l_3(y, z, \alpha)) - \tilde{x} \triangleright \lambda_3(y, z, \alpha) + \tilde{x} \triangleright ((\tilde{y}, \tilde{z}) \triangleright u) \\
& + l_3(y, z, [x, \alpha]) + \lambda_3(y, z, [x, \alpha]) - (\tilde{y}, \tilde{z}) \triangleright \lambda_1(x, \alpha) - (\tilde{y}, \tilde{z}) \triangleright (\tilde{x} \triangleright u) \\
& + l_3([x, y], z, \alpha) + \lambda_3([x, y], z, \alpha) - (\widetilde{[x, y]}, \tilde{z}) \triangleright u + c.p.(x, y, z) \\
& - [l_3(x, y, z), \alpha] - \lambda_2(l_3(x, y, z), \alpha) - \widetilde{l_3(x, y, z)} \triangleright u \\
= & -(\mathbf{D}^\mathfrak{g} \lambda)_4(x, y, z, \alpha) \\
= & 0.
\end{aligned}$$

Thus ϕ^λ is a Lie 2-algebra homomorphism. From the process above, the other hand is obvious. \blacksquare

Define $\varphi : \mathfrak{k} \oplus_\lambda \mathbb{V} \longrightarrow \mathfrak{g}$ by $\varphi = i \oplus 0$, which is a chain map. Assume that λ satisfies the condition $i_e(\mathbf{D}^\mathfrak{g} \lambda) = 0, \forall e \in \mathfrak{k}$. Then by a simple check, we find that $(\mathfrak{k} \oplus_\lambda \mathbb{V}, \mathfrak{g}, \phi^\lambda, \varphi)$ satisfies all the conditions in Proposition 3.4. Thus, we obtain:

Proposition 5.2. *Suppose that $i_e(\mathbf{D}^\mathfrak{g} \lambda) = 0, \forall e \in \mathfrak{k}$. Then $\varepsilon_\lambda = (\mathfrak{k} \oplus_\lambda \mathbb{V}, \mathfrak{g}, \phi^\lambda, \varphi)$ is a strong crossed module, where the Lie 2-algebra structure on $\mathfrak{k} \oplus_\lambda \mathbb{V}$ is given by*

$$\begin{cases} [\alpha + u, \beta + v]_\lambda &= [\alpha, \beta] + \lambda_1(\alpha, \beta), \\ [\alpha + u, \xi + m]_\lambda &= [\alpha, \xi] + \lambda_2(\alpha, \xi), \\ l_3^\lambda(\alpha + u, \beta + v, \gamma + w) &= l_3(\alpha, \beta, \gamma) + \lambda_3(\alpha, \beta, \gamma), \end{cases} \quad (28)$$

and $l_{\phi_0^\lambda(x)}$ is defined by

$$l_{\phi_0^\lambda(x)}(\alpha + u, \beta + v) = l_3(x, \alpha, \beta) + \lambda_3(x, \alpha, \beta).$$

Next, we deal with the problem of deformation. As will seen in the next proposition, we get the same strong crossed module in the isomorphic sense if λ is modified by a coboundary and an element in $\text{Im} \pi^*$.

Suppose $\lambda \in C^2(\mathfrak{g}, \mathbb{V})$ satisfying that $\mathbf{D}^\mathfrak{g} \lambda = \pi^* \theta$ for a 3-cocycle $\theta \in C^3(\mathfrak{h}, \mathbb{V})$. Then for any $R \in C^2(\mathfrak{h}, \mathbb{V})$ and $A \in C^1(\mathfrak{g}, \mathbb{V})$, we have

$$\mathbf{D}^\mathfrak{g}(\lambda + \mathbf{D}^\mathfrak{g} A + \pi^* R) = \pi^*(\theta + \mathbf{D}^\mathfrak{h} R).$$

Note that $\varepsilon_{\lambda + \mathbf{D}^\mathfrak{g} A + \pi^* R} = \varepsilon_{\lambda + \mathbf{D}^\mathfrak{g} A}$ due to $\pi^* R|_{\mathfrak{k}} = 0$. Then define $F : \mathfrak{k} \oplus_{\lambda + \mathbf{D}^\mathfrak{g} A} \mathbb{V} \longrightarrow \mathfrak{k} \oplus_\lambda \mathbb{V}$ by

$$\begin{cases} F_0(\alpha + u) &= \alpha + u + A_0(\alpha), \\ F_1(\xi + m) &= \xi + m + A_1(\xi), \\ F_2(\alpha + u, \beta + v) &= A_2(\alpha, \beta), \end{cases}$$

$G = \text{Id} : \mathfrak{g} \longrightarrow \mathfrak{g}$, and $\tau : \mathfrak{g}_0 \wedge (\mathfrak{k} \oplus_{\lambda + \mathbf{D}^\mathfrak{g} A} \mathbb{V})_0 \longrightarrow (\mathfrak{k} \oplus_\lambda \mathbb{V})_1$ by $\tau(x, \alpha + u) = A_2(x, \alpha)$.

Proposition 5.3. *The map (F, G, τ) is an isomorphism from $\varepsilon_{\lambda + \mathbf{D}^\mathfrak{g} A + \pi^* R}$ to ε_λ . We call it a gauge transformation.*

Proof. Firstly, F is a Lie 2-algebra homomorphism due to the formulation of $\mathbf{D}^\mathfrak{g} A$. Indeed, the condition $F_0 \circ d^{\lambda + \mathbf{D}^\mathfrak{g} A} = d^\lambda \circ F_1$ follows from

$$\begin{aligned}
& (F_0 \circ d^{\lambda + \mathbf{D}^\mathfrak{g} A} - d^\lambda \circ F_1)(\xi + m) \\
= & F_0(d\xi + d^\mathbb{V} m + (\lambda_0 + (\mathbf{D}^\mathfrak{g} A)_0)(\xi)) - d^\lambda(\xi + m + A_1(\xi)) \\
= & d\xi + d^\mathbb{V} m + \lambda_0(\xi) + (\mathbf{D}^\mathfrak{g} A)_0(\xi) + A_0(d\xi) - d\xi - d^\mathbb{V}(m + A_1(\xi)) - \lambda_0(\xi) \\
= & (\mathbf{D}^\mathfrak{g} A)_0(\xi) + A_0(d\xi) - d^\mathbb{V} A_1(\xi) \\
= & 0.
\end{aligned}$$

Then, we shall verify the coherence condition, since the other two conditions of homomorphism are similar to get. By a straightforward calculation, we have

$$\begin{aligned}
& [F_0(\alpha + u), F_2(\beta + v, \gamma + w)]_\lambda - F_2([\alpha + u, \beta + v]_{\lambda + D^g A}, \gamma + w) + c.p. \\
& + l_3^\lambda(F_0(\alpha + u), F_0(\beta + v), F_0(\gamma + w)) - F_1 l_3^{\lambda + D^g A}(\alpha + u, \beta + v, \gamma + w) \\
= & -A_2([\alpha, \beta], \gamma) + c.p. + l_3(\alpha, \beta, \gamma) - l_3(\alpha, \beta, \gamma) - (D^g A)_3(\alpha, \beta, \gamma) - A_1 l_3(\alpha, \beta, \gamma) \\
= & 0.
\end{aligned}$$

Next, we get $\varphi \circ F = G \circ \varphi$ by definition. Finally, similar to the above procedure, it is direct to check that

$$((G_0, F_0), (G_1, F_1), (G_2, \tau, F_2)) : \mathfrak{g} \triangleright_{\lambda + D^g A} (\mathfrak{k} \oplus_{\lambda + D^g A} \mathbb{V}) \longrightarrow \mathfrak{g} \triangleright_\lambda (\mathfrak{k} \oplus_\lambda \mathbb{V})$$

is a Lie 2-algebra homomorphism. Thus, (F, G, τ) is a morphism from $\varepsilon_{\lambda + D^g A}$ to ε_λ . Furthermore, note that F and G are bijections as chain maps and $\varepsilon_{\lambda + D^g A + \pi^* R} = \varepsilon_{\lambda + D^g A}$. We conclude that $\varepsilon_{\lambda + D^g A + \pi^* R}$ and ε_λ are isomorphic. ■

Generally, for a 3-cocycle $\theta \in C^3(\mathfrak{h}, \mathbb{V})$, the 3-cochain $\pi^* \theta$ is not necessarily to be a 3-coboundary. Nevertheless, consider the short exact sequence

$$0 \rightarrow \ker \pi \hookrightarrow \mathfrak{F}(\mathfrak{h}) \xrightarrow{\pi} \mathfrak{h} \rightarrow 0,$$

where $\mathfrak{F}(\mathfrak{h})$ is the free Lie 2-algebra ([17]) generated by the underlying 2-vector space of \mathfrak{h} and π is the canonical projection. For a 3-cocycle $\theta \in C^3(\mathfrak{h}, \mathbb{V})$, since the second and third cohomology of any free Lie 2-algebra are trivial¹, there exists a 2-cochain $\lambda \in C^2(\mathfrak{F}(\mathfrak{h}), \mathbb{V})$ such that $D^{\mathfrak{F}(\mathfrak{h})} \lambda = \pi^* \theta$ and for different 2-cochains λ, λ' satisfying it, we have $[\lambda - \lambda'] = 0$. By Proposition 5.3, we have:

Corollary 5.4. *For any $[\theta] \in H^3(\mathfrak{h}, \mathbb{V})$, we get a class of crossed modules differing from each other by a gauge transformation*

$$\{\varepsilon_\lambda; D^{\mathfrak{F}(\mathfrak{h})} \lambda = \pi^* \theta', [\theta'] = [\theta] \in H^3(\mathfrak{h}, \mathbb{V})\}.$$

5.2 Classification

A crossed module $(\mathfrak{m}, \mathfrak{g}, \phi, \varphi, \sigma)$ can yield a 4-term exact sequence of 2-vector spaces

$$0 \longrightarrow \mathbb{V} \xrightarrow{i} \mathfrak{m} \xrightarrow{\varphi} \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \longrightarrow 0, \quad (29)$$

where $\mathbb{V} \triangleq \ker \varphi$, $\mathfrak{h} \triangleq \text{coker} \varphi$, and i, π are the canonical inclusion and projection. By (i), (ii) of Definition 3.2, \mathbb{V} is in the center of \mathfrak{m} . However, it needs some extra conditions to ensure that \mathfrak{h} is a Lie 2-algebra and there exists an induced action of \mathfrak{h} on \mathbb{V} .

Lemma 5.5. *For a crossed module of Lie 2-algebras $(\mathfrak{m}, \mathfrak{g}, \phi, \varphi, \sigma)$, we have*

- (1) $\mathfrak{h} = \mathfrak{g}/\text{Im} \varphi$ is a quotient Lie 2-algebra of \mathfrak{g} if $\text{Im} \sigma \subset \text{Im} \varphi_1$;
- (2) the action of \mathfrak{g} on \mathfrak{m} induces an \mathfrak{h} -module structure on \mathbb{V} if $\text{Im} \sigma \subset \text{Im} \varphi_1$ and $\sigma(\ker \varphi_0 \wedge \mathfrak{g}_0) = 0$.

In particular, \mathbb{V} becomes a module of Lie 2-algebra \mathfrak{h} if the crossed module is strong.

¹It is due to that the cohomology of Lie 2-algebra is “operadic” in nature and the operadic cohomology (degree ≥ 2) vanishes on frees. See [18].

Proof. In order to prove $\text{Im}\varphi$ is an ideal, we need to verify $l_2(\text{Im}\varphi \wedge \mathfrak{g}) \subset \text{Im}\varphi$, $l_3(\text{Im}\varphi_0 \wedge \mathfrak{g}_0 \wedge \mathfrak{g}_0) \subset \text{Im}\varphi_1$. Following that $\Pi = Id + \sigma + \varphi$ is a Lie 2-algebra homomorphism, we have

$$\begin{cases} \varphi_0(x \triangleright \alpha) - [x, \varphi_0(\alpha)] &= d\sigma(x, \alpha), & \forall x \in \mathfrak{g}_0, \alpha \in \mathfrak{m}_0, \\ \varphi_1(a \triangleright \alpha) - [a, \varphi_0(\alpha)] &= \sigma(da, \alpha), & \forall a \in \mathfrak{g}_1, \alpha \in \mathfrak{m}_0, \\ \varphi_1(x \triangleright \xi) - [x, \varphi_1(\xi)] &= \sigma(x, \tilde{d}\xi), & \forall x \in \mathfrak{g}_0, \xi \in \mathfrak{m}_1, \end{cases} \quad (30)$$

which leads to $l_2(\text{Im}\varphi \wedge \mathfrak{g}) \subset \text{Im}\varphi$ since $\text{Im}\sigma \subset \text{Im}\varphi_1$. Moreover, consider the coherence law in the definition of homomorphism, for any $x, y \in \mathfrak{g}_0, \alpha \in \mathfrak{m}_0$,

$$[\sigma(y, \alpha), x] + [\sigma(\alpha, x), y] + \sigma([x, y], \alpha) + \sigma(y \triangleright \alpha, x) - \sigma(x \triangleright \alpha, y) = l_3(x, y, \varphi_0\alpha) - \varphi_1 L_3(x, y, \alpha),$$

which implies that $l_3(x, y, \varphi_0\alpha) \in \text{Im}\varphi_1$, i.e., $l_3(\text{Im}\varphi_0 \wedge \mathfrak{g}_0 \wedge \mathfrak{g}_0) \subset \text{Im}\varphi_1$.

To prove (2), firstly, by equalities (30) and $\sigma(\ker \varphi_0 \wedge \mathfrak{g}_0) = 0$, we get $\phi_i : \mathfrak{g}_i \longrightarrow \text{End}_i(\ker \varphi)$, $i = 0, 1$. Coupled with the coherence law above, it is clear that

$$(x, y) \triangleright \beta = -L_3(x, y, \beta) \in \ker \varphi_1, \quad \forall \beta \in \ker \varphi_0, x, y \in \mathfrak{g}_0.$$

Namely, $\phi_2 : \wedge^2 \mathfrak{g}_0 \longrightarrow \text{End}_1(\ker \varphi)$. Thus we can define $\tilde{\phi} = \phi \circ s : \mathfrak{h} \longrightarrow \text{End}(\ker \varphi)$, where $s : \mathfrak{h} \longrightarrow \mathfrak{g}$ is a section of π .

Next, we prove $\tilde{\phi}$ is independent of the section s . Let s' be another section of π , then $\text{Im}(s - s') \in \ker \pi = \text{Im}\varphi$. We shall check $\phi \circ (s - s') = 0$, which is equivalent to show the homomorphism $\mathfrak{g} \xrightarrow{\phi} \text{End}(\ker \varphi)$ vanishes when restricting on $\text{Im}\varphi$. Depending on (i), (iii) of Definition 3.2, we have

$$\varphi(\alpha) \triangleright \beta = -\varphi(\beta) \triangleright \alpha = 0, \quad \forall \beta \in \ker \varphi, \alpha \in \mathfrak{m}.$$

And, for any $\beta \in \ker \varphi_0, \alpha \in \mathfrak{m}_0, x \in \mathfrak{g}_0$,

$$(x, \varphi_0\alpha) \triangleright \beta = -(x, \varphi_0\beta) \triangleright \alpha - \sigma(x, \beta) \triangleright \alpha - \sigma(x, \alpha) \triangleright \beta = -\sigma(x, \alpha) \triangleright \beta,$$

which vanishes since $\text{Im}\sigma \subset \text{Im}\varphi_1$ and $\beta \in \ker \varphi_0$.

At last, since $[sx, sy] \triangleright \alpha = s[x, y] \triangleright \alpha, \forall x, y \in \mathfrak{h}, \alpha \in \ker \varphi$, it is direct to check $\tilde{\phi}$ is a Lie 2-algebra homomorphism. Therefore, the requirements $\text{Im}\sigma \subset \text{Im}\varphi_1$ and $\sigma(\ker \varphi_0 \wedge \mathfrak{g}_0) = 0$ make sure that the action of \mathfrak{g} on \mathfrak{m} can induce an action of \mathfrak{h} on $\ker \varphi$. ■

Note that for a strong crossed module $(\mathfrak{m}, \mathfrak{g}, \phi, \varphi)$, there exists an exact sequence (29) satisfying that all the homomorphisms are strong. Denote by $\mathcal{C}(\mathfrak{h}, \mathbb{V})$ the set of strong crossed modules with respect to fixed kernel \mathbb{V} and fixed cokernel \mathfrak{h} and fixed action of \mathfrak{h} on \mathbb{V} .

Example 5.6. Consider the set in Proposition 5.4 denoted by $\mathcal{C}(\mathfrak{h}, \mathbb{V})_{[\theta]}$. In fact,

$$\mathcal{C}(\mathfrak{h}, \mathbb{V})_{[\theta]} = \{\varepsilon_\lambda; D^{\tilde{\mathfrak{F}}(\mathfrak{h})}\lambda = \pi^*\theta', [\theta'] = [\theta] \in H^3(\mathfrak{h}, \mathbb{V})\}$$

is a subset of $\mathcal{C}(\mathfrak{h}, \mathbb{V})$.

Definition 5.7. Let $(\mathfrak{m}, \mathfrak{g}, \phi, \varphi)$ and $(\mathfrak{m}', \mathfrak{g}', \phi', \varphi')$ be two strong crossed modules in $\mathcal{C}(\mathfrak{h}, \mathbb{V})$, a strong map between them is a strong morphism of crossed modules (F, G) such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{V} & \xrightarrow{i} & \mathfrak{m} & \xrightarrow{\varphi} & \mathfrak{g} & \xrightarrow{\pi} & \mathfrak{h} & \longrightarrow & 0 \\ & & Id \downarrow & & F \downarrow & & G \downarrow & & Id \downarrow & & \\ 0 & \longrightarrow & \mathbb{V} & \xrightarrow{i'} & \mathfrak{m}' & \xrightarrow{\varphi'} & \mathfrak{g}' & \xrightarrow{\pi'} & \mathfrak{h} & \longrightarrow & 0, \end{array} \quad (31)$$

is commutative.

Definition 5.8. For two crossed module $\varepsilon, \varepsilon' \in \mathcal{C}(\mathfrak{h}, \mathbb{V})$, we define $\varepsilon \sim \varepsilon'$ if there exist two crossed module $\varepsilon_\lambda, \varepsilon_{\lambda'} \in \mathcal{C}(\mathfrak{h}, \mathbb{V})_{[\theta]}$ for a 3-cocycle $\theta \in C^3(\mathfrak{h}, \mathbb{V})$ such that the diagram

$$\begin{array}{ccc} & \varepsilon_\lambda \cong \varepsilon_{\lambda'} & \\ f \swarrow & & \searrow f' \\ \varepsilon & \sim & \varepsilon' \end{array} \quad (32)$$

holds, where f, f' are strong maps in $\mathcal{C}(\mathfrak{h}, \mathbb{V})$ and \cong is the gauge transformation between them.

Since the composition of gauge transformations is also a gauge transformation, it is evident that \sim is an equivalence relation.

Remark 5.9. The linear map $\varphi : \mathfrak{m} \rightarrow \mathfrak{g}$ induces an action groupoid $\mathfrak{g} \times \mathfrak{m} \rightrightarrows \mathfrak{g}$ with the abelian group structure on \mathfrak{m} , where the source, target and inclusion maps are $s(x, \alpha) = x, t(x, \alpha) = x + \varphi(\alpha), i(x) = (x, 0)$. Then φ has fixed kernel and cokernel means that the groupoid has fixed isotropy subgroup and orbit space. In this viewpoint, the relation between ε and ε' defined by (32) is in fact a generalized map ([19]) between them.

Theorem 5.10. For a Lie 2-algebra \mathfrak{h} and an \mathfrak{h} -module \mathbb{V} , there exists a canonical bijection

$$\mathcal{C}(\mathfrak{h}, \mathbb{V}) / \sim \xrightarrow{\cong} H^3(\mathfrak{h}, \mathbb{V}),$$

where \sim is the equivalence relation defined in Definition 5.8.

We divide the proof into four steps. For simplicity, we denote by the same notations $d, [\cdot, \cdot]$ and l_3 for different Lie 2-algebras except when emphasis is needed, which will not cause any confusion. In the rest of this section, we will always suppose $x, y, z \in \mathfrak{h}_0$ and $a, b \in \mathfrak{h}_1$.

Step 1: Construct a map $\mu : \mathcal{C}(\mathfrak{h}, \mathbb{V}) \rightarrow H^3(\mathfrak{h}, \mathbb{V})$. To a crossed module $\varepsilon = (\mathfrak{m}, \mathfrak{g}, \phi, \varphi) \in \mathcal{C}(\mathfrak{h}, \mathbb{V})$, choose linear sections $s = (s_0, s_1) : \mathfrak{h} \rightarrow \mathfrak{g}$, $\pi s = Id$ and $q = (q_0, q_1) : \text{Im} \varphi \rightarrow \mathfrak{m}$, $\varphi q = Id$. Since $ds_1(a) - s_0 d(a) \in \ker \pi_0 = \text{Im} \varphi_0$, we can take

$$\lambda_0(a) = q_0(ds_1(a) - s_0 d(a)).$$

Similarly, take

$$\begin{cases} \lambda_1(x, y) &= q_0([s_0(x), s_0(y)] - s_0[x, y]), \\ \lambda_2(x, a) &= q_1([s_0(x), s_1(a)] - s_1[x, a]), \\ \lambda_3(x, y, z) &= q_1(l_3(s_0(x), s_0(y), s_0(z)) - s_1 l_3(x, y, z)). \end{cases}$$

Note that $\lambda_\varepsilon = \sum_{i=0}^3 \lambda_i$ satisfies $\pi^* \lambda_\varepsilon \in C^2(\mathfrak{g}, \mathfrak{m})$. It is reasonable to define

$$\theta_\varepsilon = s^* D^g(\pi^* \lambda_\varepsilon). \quad (33)$$

Lemma 5.11. With the above notations, we have

- (1) $\varphi \circ \theta_\varepsilon = 0$, that is, $\theta_\varepsilon \in C^3(\mathfrak{h}, \mathbb{V})$.
- (2) $D^h \theta_\varepsilon = 0$.

Proof. According to (10), we shall check that $\varphi_0 \circ \theta_{\varepsilon_j} = 0$ for $j=0,2$ and $\varphi_1 \circ \theta_{\varepsilon_j} = 0$ for $j=1,3,4$. The cases of $j = 0, 1, 2, 3$ are quite straightforward, so we omit the details. For the case of $j = 4$, since $\Pi = Id + \varphi$ is a strong homomorphism, we have

$$l_3(s_0(x_{\sigma_1}), s_0(x_{\sigma_2}), \varphi_0 \lambda_1(x_{\sigma_3}, x_{\sigma_4})) = -\varphi_1((s_0(x_{\sigma_1}), s_0(x_{\sigma_2})) \triangleright \lambda_1(x_{\sigma_3}, x_{\sigma_4})). \quad (34)$$

Taking into account (10), (34) and the coherence laws of l_2, l_3 of Lie 2-algebras \mathfrak{g} and \mathfrak{h} , we have

$$\begin{aligned}
& \varphi_1(\theta_{\varepsilon 4}(x_1, x_2, x_3, x_4)) \\
= & - \sum_{\sigma} (-1)^{\sigma} l_3(s_0(x_{\sigma_1}), s_0(x_{\sigma_2}), [s_0(x_{\sigma_3}), s_0(x_{\sigma_4})] - s_0[x_{\sigma_3}, x_{\sigma_4}]) \\
& - \sum_{\tau} (-1)^{\tau} \{ [s_0(x_{\tau_4}), s_1 l_3(x_{\tau_1}, x_{\tau_2}, x_{\tau_3})] - s_1[x_{\tau_4}, l_3(x_{\tau_1}, x_{\tau_2}, x_{\tau_3})] \} \\
& + \sum_{i=1}^4 (-1)^{i+1} [s_0(x_i), l_3(s_0(x_1), \dots, s_0(\widehat{x_i}), \dots, s_0(x_4)) - s_1 l_3(x_1, \dots, \widehat{x_i}, \dots, x_4)] \\
& + \sum_{i < j} (-1)^{i+j} \{ l_3(s_0[x_i, x_j], s_0(x_1), \dots, s_0(\widehat{x_i}), \dots, s_0(\widehat{x_j}), \dots, s_0(x_4)) \\
& - s_1 l_3([x_i, x_j], x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_4) \} \\
= & - \sum_{\sigma} (-1)^{\sigma} l_3(s_0(x_{\sigma_1}), s_0(x_{\sigma_2}), [s_0(x_{\sigma_3}), s_0(x_{\sigma_4})]) \\
& + \sum_{i=1}^4 (-1)^{i+1} [s_0(x_i), l_3(s_0(x_1), \dots, s_0(\widehat{x_i}), \dots, s_0(x_4))] \\
& + \sum_{\tau} (-1)^{\tau} s_1[x_{\tau_4}, l_3(x_{\tau_1}, x_{\tau_2}, x_{\tau_3})] + \sum_{i < j} (-1)^{i+j} s_1 l_3([x_i, x_j], x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_4) \\
= & 0.
\end{aligned}$$

Therefore, $\text{Im} \theta_{\varepsilon} \subset \mathbb{V}$.

Note that if only $\text{Im} \lambda_{\varepsilon} \subset \mathbb{V}$, the definition of θ_{ε} would be read as $\theta_{\varepsilon} = s^* \pi^* D^{\mathfrak{h}}(\lambda_{\varepsilon}) = D^{\mathfrak{h}} \lambda_{\varepsilon}$ and hence would give $D^{\mathfrak{h}} \theta_{\varepsilon} = 0$. In our situation, straightforward calculations show that $D^{\mathfrak{h}} \theta_{\varepsilon}$ still vanishes. Explicitly, by the definition of $D^{\mathfrak{h}}$, we have

$$\left\{ \begin{array}{ll} (D^{\mathfrak{h}} \theta_{\varepsilon})_0 &= \hat{d} \theta_0 + \widehat{d^{\mathbb{V}}} \theta_1 & \in \text{Hom}(\odot^2 \mathfrak{g}_1, V_0), \\ (D^{\mathfrak{h}} \theta_{\varepsilon})_1 &= d_{\phi}^{(1,0)} \theta_0 + \hat{d} \theta_2 + \widehat{d^{\mathbb{V}}} \theta_3 & \in \text{Hom}(\wedge^2 \mathfrak{g}_0 \wedge \mathfrak{g}_1, V_0), \\ (D^{\mathfrak{h}} \theta_{\varepsilon})_2 &= d_{\phi}^{(0,1)} \theta_0 + d_{\phi}^{(1,0)} \theta_1 + \hat{d} \theta_3 & \in \text{Hom}(\mathfrak{g}_0 \wedge \odot^2 \mathfrak{g}_1, V_1), \\ (D^{\mathfrak{h}} \theta_{\varepsilon})_3 &= d_{\phi_2} \theta_0 + d_{l_3} \theta_1 + d_{\phi}^{(0,1)} \theta_2 + d_{\phi}^{(1,0)} \theta_3 + \hat{d} \theta_4 & \in \text{Hom}(\wedge^3 \mathfrak{g}_0 \wedge \mathfrak{g}_1, V_1), \\ (D^{\mathfrak{h}} \theta_{\varepsilon})_4 &= d_{l_3} \theta_0 + d_{\phi}^{(1,0)} \theta_2 + \widehat{d^{\mathbb{V}}} \theta_4 & \in \text{Hom}(\wedge^4 \mathfrak{g}_0, V_0), \\ (D^{\mathfrak{h}} \theta_{\varepsilon})_5 &= d_{\phi_2} \theta_2 + d_{l_3} \theta_3 + d_{\phi}^{(1,0)} \theta_4 & \in \text{Hom}(\wedge^5 \mathfrak{g}_0, V_1), \end{array} \right.$$

where $\theta_j = \theta_{\varepsilon j} = (s^* D^{\mathfrak{g}}(\pi^* \lambda_{\varepsilon}))_j$ as in (10). By direct calculations, we have

$$\begin{aligned}
(D^{\mathfrak{h}} \theta_{\varepsilon})_0(a, b) &= -s_0(da) \triangleright \lambda_0(b) + \lambda_0[da, b] - \lambda_1(da, db) + d^{\mathfrak{m}} \lambda_2(da, b) \\
&\quad - s_0(db) \triangleright \lambda_0(a) + \lambda_0[db, a] - \lambda_1(db, da) + d^{\mathfrak{m}} \lambda_2(db, a) \\
&\quad + d^{\mathbb{V}}(s_1(a) \triangleright \lambda_0(b) + s_1(b) \triangleright \lambda_0(a) - \lambda_2(da, b) + \lambda_2(a, db)) \\
&= \varphi_0 \lambda_0(a) \triangleright \lambda_0(b) + \varphi_0 \lambda_0(b) \triangleright \lambda_0(a),
\end{aligned}$$

which vanishes due to condition (i) of Definition 3.2. Similarly, we can deduce that $D^{\mathfrak{h}} \theta_{\varepsilon} = 0$. ■

As has been already demonstrated, to each strong crossed module $\varepsilon = (\mathfrak{m}, \mathfrak{g}, \phi, \varphi) \in \mathcal{C}(\mathfrak{h}, \mathbb{V})$, one can obtain a 3-cocycle $\theta_{\varepsilon} \in C^3(\mathfrak{h}, \mathbb{V})$. Then, define $\mu(\varepsilon) = [\theta_{\varepsilon}]$.

Step 2: We shall prove the canonical property of the map μ . Namely, μ is independent of the choices made of sections s and q . Here, for future reference, we also prove that if there is a strong map $\varepsilon \rightarrow \varepsilon'$ in $\mathcal{C}(\mathfrak{h}, V)$, then θ_{ε} equals $\theta_{\varepsilon'}$ in $H^3(\mathfrak{h}, \mathbb{V})$.

Lemma 5.12. θ_ε is independent of the choice of section s .

Proof. Suppose that $\bar{s} = (\bar{s}_0, \bar{s}_1)$ is another section of π and let $\bar{\theta}_\varepsilon$ be the 3-cocycle defined using \bar{s} instead of s . We need to prove that $\bar{\theta}_\varepsilon$ coincides with θ_ε in $H^3(\mathfrak{h}, \mathbb{V})$.

Since \bar{s} and s are both sections of π , there exist two linear maps $t_i : \mathfrak{h}_i \longrightarrow \mathfrak{m}_i$ with $s_i - \bar{s}_i = \varphi_i \circ t_i$. Construct four maps as follows, for any $x, y, z \in \mathfrak{h}_0, a \in \mathfrak{h}_1$,

$$\left\{ \begin{array}{lcl} B_0(a) & = & d^m t_1(a) - t_0(da), \\ B_1(x, y) & = & \bar{s}_0(x) \triangleright t_0(y) - s_0(y) \triangleright t_0(x) - t_0[x, y], \\ B_2(x, a) & = & \bar{s}_0(x) \triangleright t_1(a) - s_1(a) \triangleright t_0(x) - t_1[x, a], \\ B_3(x, y, z) & = & -(s_0(x), s_0(y)) \triangleright t_0(z) - (\bar{s}_0(x), \bar{s}_0(y)) \triangleright t_0(z) + (s_0(x), \bar{s}_0(y)) \triangleright t_0(z) \\ & & - (\bar{s}_0(y), s_0(z)) \triangleright t_0(x) - (\bar{s}_0(z), s_0(x)) \triangleright t_0(y). \end{array} \right. \quad (35)$$

Since $\varphi_0 d = d^m \varphi_1$, it is obvious that

$$\begin{aligned} \varphi_0(B_0(a)) &= ds_1(a) - d\bar{s}_1(a) - s_0(da) + \bar{s}_0(da) \\ &= \varphi_0(\lambda_0(a) - \bar{\lambda}_0(a)), \end{aligned}$$

which implies that

$$\lambda_0 - \bar{\lambda}_0 - B_0 \in \text{Hom}(\mathfrak{h}_1, V_0).$$

Similarly, relying on $\Pi = Id + \varphi$ is a strong homomorphism, we can deduce that $\lambda - \bar{\lambda} - B \in C^2(\mathfrak{h}, \mathbb{V})$, where $B = \sum_{i=0}^3 B_i$. Furthermore, we claim that

$$\theta_\varepsilon - \bar{\theta}_\varepsilon = D^h(\lambda - \bar{\lambda} - B).$$

By straightforward computations, we have

$$\begin{aligned} (\theta_\varepsilon - \bar{\theta}_\varepsilon)_0(x, a) &= s_0(x) \triangleright (\lambda_0 - \bar{\lambda}_0)(a) - (\lambda_0 - \bar{\lambda}_0)[x, a] + \varphi_0 t_0(x) \triangleright \bar{\lambda}_0(a) \\ &\quad + (\lambda_1 - \bar{\lambda}_1)(x, da) - d^m(\lambda_2 - \bar{\lambda}_2)(x, a), \end{aligned} \quad (36)$$

and

$$\begin{aligned} (D^h(\lambda - \bar{\lambda} - B))_0(x, a) &= s_0(x) \triangleright (\lambda_0 - \bar{\lambda}_0 - B_0)(a) - (\lambda_0 - \bar{\lambda}_0 - B_0)[x, a] \\ &\quad + (\lambda_1 - \bar{\lambda}_1 - B_1)(x, da) - d^m(\lambda_2 - \bar{\lambda}_2 - B_2)(x, a). \end{aligned} \quad (37)$$

Then substituting B_i by the right hand sides of (35) and taking into account condition (i) of Definition 3.2, we get

$$\begin{aligned} &(36) - (37) \\ &= s_0(x) \triangleright (d^m t_1(a) - t_0(da)) - d^m t_1[x, a] + t_0[x, da] + (\bar{s}_0(da) - d\bar{s}_1(a)) \triangleright t_0(x) \\ &\quad + \bar{s}_0(x) \triangleright t_0(da) - s_0(da) \triangleright t_0(x) - t_0[x, da] - d^m(\bar{s}_0(x) \triangleright t_1(a) - s_1(a) \triangleright t_0(x) - t_1[x, a]) \\ &= \varphi_0 t_0(x) \triangleright d^m t_1(a) + d^m(\varphi_1 t_1(a) \triangleright t_0(x)) - \varphi_0 t_0(x) \triangleright t_0(da) - \varphi_0 t_0(da) \triangleright t_0(x) \\ &= 0. \end{aligned}$$

Similarly, we can verify that $\theta_\varepsilon - \bar{\theta}_\varepsilon = D^h(\lambda - \bar{\lambda} - B)$, which proves that the class of θ_ε is independent of the section s . ■

Lemma 5.13. θ_ε is independent of the choice of section q and a strong map $\varepsilon \longrightarrow \varepsilon'$.

Proof. Consider a strong map $(F, G) : \varepsilon \longrightarrow \varepsilon'$ as in Definition 5.7. Let $s : \mathfrak{h} \longrightarrow \mathfrak{g}$ and $q : \text{Im}\varphi \longrightarrow \mathfrak{m}$ be sections of π and φ and let $s' : \mathfrak{h}' \longrightarrow \mathfrak{g}'$ and $q' : \text{Im}\varphi' \longrightarrow \mathfrak{m}'$ be sections of π' and φ' . Since $\pi'(Gs) = \pi(s) = Id$, we get another section $Gs = (G_1s_1, G_0s_0)$ of π' . Taking into account Lemma 5.12, we choose $s' = Gs$. Set

$$\begin{cases} B_0(a) &= (F_0q_0 - q'_0G_0)(ds_1(a) - s_0d(a)), \\ B_1(x, y) &= (F_0q_0 - q'_0G_0)([s_0(x), s_0(y)] - s_0[x, y]), \\ B_2(x, a) &= (F_1q_1 - q'_1G_1)([s_0(x), s_1(a)] - s_1[x, a]), \\ B_3(x, y, z) &= (F_1q_1 - q'_1G_1)(l_3(s_0(x), s_0(y), s_0(z)) - s_1l_3(x, y, z)). \end{cases}$$

Noticing that $\varphi'F = G\varphi$, it is obvious that $B = \sum_{i=0}^3 B_i \in C^2(\mathfrak{h}, \mathbb{V})$. Furthermore, relying on the properties of (F, G) , we obtain

$$\begin{aligned} & (\theta_{\varepsilon_0} - \theta_{\varepsilon'_0})(x, a) \\ &= F_0(\theta_{\varepsilon_0}(x, a)) - \theta_{\varepsilon'_0}(x, a) \\ &= F_0(s_0(x) \triangleright q_0(ds_1(a) - s_0(da)) - q_0(ds_1[x, a] - s_0d[x, a]) \\ &\quad + q_0([s_0(x), s_0(da)] - s_0[x, da]) - d^{\mathfrak{m}}q_1([s_0(x), s_1(a)] - s_1[x, a])) \\ &\quad - G_0s_0(x) \triangleright' q'_0(d'G_1s_1(a) - G_0s_0(da)) + q'_0(d'G_1s_1[x, a] - G_0s_0d[x, a]) \\ &\quad - q'_0([G_0s_0(x), G_0s_0(da)] - G_0s_0[x, da]) + d^{\mathfrak{m}'}q'_1([G_0s_0(x), G_1s_1(a)] - G_1s_1[x, a]) \\ &= G_0s_0(x) \triangleright' (F_0q_0 - q'_0G_0)(ds_1(a) - s_0(da)) - (F_0q_0 - q'_0G_0)(ds_1[x, a] - s_0d[x, a]) \\ &\quad + (F_0q_0 - q'_0G_0)([s_0(x), s_0(da)] - s_0[x, da]) - d^{\mathfrak{m}'}(F_1q_1 - q'_1G_1)([s_0(x), s_1(a)] - s_1[x, a]) \\ &= (d_{\phi'}^{(1,0)}B_0 + \hat{d}B_1 + \hat{d}^{\mathfrak{m}'}B_2)(x, a). \end{aligned}$$

Similarly, considering the fact that ϕ and ϕ' induce the same \mathfrak{h} -module structure on \mathbb{V} , we get

$$\theta_{\varepsilon} - \theta_{\varepsilon'} = D^{\mathfrak{h}}B.$$

This finishes the proof. \blacksquare

Step 3: We show μ is a surjection, which follows from Corollary 5.4 and the following lemma.

Lemma 5.14. *For any $[\theta] \in H^3(\mathfrak{h}, \mathbb{V})$, we have $\mu(\mathcal{C}(\mathfrak{h}, \mathbb{V})_{[\theta]}) = [\theta]$.*

Proof. For a crossed module $\varepsilon_{\lambda} = (\ker \pi \oplus_{\lambda} \mathbb{V}, \mathfrak{F}(\mathfrak{h}), \phi^{\lambda}, \varphi) \in \mathcal{C}(\mathfrak{h}, \mathbb{V})_{[\theta]}$, consider the complex

$$\varepsilon_{\lambda} : 0 \rightarrow \mathbb{V} \xrightarrow{i} \ker \pi \oplus_{\lambda} \mathbb{V} \xrightarrow{\varphi} \mathfrak{F}(\mathfrak{h}) \xrightarrow{\pi} \mathfrak{h} \rightarrow 0.$$

Choosing any section s of π and defining section q of φ on $\text{Im}\varphi$ by $q(\alpha) = (\alpha, 0), \forall \alpha \in \ker \pi$, we get a 3-cocycle $\theta_{\varepsilon_{\lambda}} \in C^3(\mathfrak{h}, \mathbb{V})$. We claim that $\theta_{\varepsilon_{\lambda}} + D^{\mathfrak{h}}s^*\lambda = \theta$, which implies that $\mu(\varepsilon_{\lambda}) = [\theta_{\varepsilon_{\lambda}}] = [\theta]$. Actually, by (33) and (10), we have

$$\begin{aligned} (\theta_{\varepsilon_{\lambda}})_0(x, a) &= s_0x \triangleright_{\lambda} (ds_1a - s_0da) - ds_1[x, a] + s_0d[x, a] \\ &\quad + [s_0x, s_0da] - s_0[x, da] - d^{\lambda}([s_0x, s_1a] - s_1[x, a]) \\ &= \lambda_1(s_0x, ds_1a - s_0da) - \lambda_0([s_0x, s_1a] - s_1[x, a]), \end{aligned}$$

and

$$(D^{\mathfrak{h}}s^*\lambda)_0(x, a) = x \triangleright \lambda_0(s_1a) - \lambda_0s_1[x, a] + \lambda_1(s_0x, s_0da) - d^{\mathbb{V}}\lambda_2(s_0x, s_1a).$$

Adding them together, we get

$$\begin{aligned}
(\theta_{\varepsilon_\lambda} + D^h s^* \lambda)_0(x, a) &= x \triangleright \lambda_0(s_1 a) - \lambda_0[s_0 x, s_1 a] + \lambda_1(s_0 x, ds_1 a) - d^\vee \lambda_2(s_0 x, s_1 a) \\
&= (D^{\mathfrak{F}(\mathfrak{h})} \lambda)_0(s_0 x, s_1 a) = (\pi^* \theta)_0(s_0 x, s_1 a) \\
&= \theta_0(x, a).
\end{aligned}$$

Likewise, we can obtain $\theta_{\varepsilon_\lambda} + D^h s^* \lambda = \theta$. This finishes the proof. \blacksquare

Step 4: We shall prove for two crossed modules $\varepsilon, \varepsilon' \in \mathcal{C}(\mathfrak{h}, \mathbb{V})$, $\mu(\varepsilon) = \mu(\varepsilon')$ iff $\varepsilon \sim \varepsilon'$. Then, the map $\mu : \mathcal{C}(\mathfrak{h}, \mathbb{V}) \longrightarrow H^3(\mathfrak{h}, \mathbb{V})$ induces a bijection between $\mathcal{C}(\mathfrak{h}, \mathbb{V})/\sim$ and $H^3(\mathfrak{h}, \mathbb{V})$.

A direct consequence of Lemma 5.13 and 5.14 is:

Corollary 5.15. *If $\varepsilon \sim \varepsilon'$, we have $\mu(\varepsilon) = \mu(\varepsilon')$.*

Proposition 5.16. *For a 3-cocycle $\theta \in C^3(\mathfrak{h}, \mathbb{V})$, $\mu(\varepsilon) = [\theta]$ if and only if there exists a crossed module $\varepsilon_\lambda \in \mathcal{C}(\mathfrak{h}, \mathbb{V})_{[\theta]}$ and a strong map $(F, G) : \varepsilon_\lambda \longrightarrow \varepsilon$. That is, if $\mu(\varepsilon) = \mu(\varepsilon')$, then $\varepsilon \sim \varepsilon'$.*

Proof. For such a crossed module $\varepsilon = (\mathfrak{m}, \mathfrak{g}, \phi, \varphi) \in \mathcal{C}(\mathfrak{h}, \mathbb{V})$, choosing sections $s : \mathfrak{h} \longrightarrow \mathfrak{g}$ and $q : \text{Im } \varphi \longrightarrow \mathfrak{m}$ of π' and φ respectively, we can construct λ_ε and then a 3-cocycle $\theta_\varepsilon \in C^3(\mathfrak{h}, \mathbb{V})$ as in (33) such that $[\theta_\varepsilon] = [\theta]$. Let $G : \mathfrak{F}(\mathfrak{h}) \longrightarrow \mathfrak{g}$ be the strong Lie 2-algebra homomorphism induced by s (the property of free Lie 2-algebras). Define linear maps $\psi = (\psi_0, \psi_1) : \mathfrak{F}(\mathfrak{h}) \longrightarrow \mathfrak{m}$ by $\psi(\bar{x}) = q(G\bar{x} - s\pi\bar{x})$, $\forall \bar{x} \in \mathfrak{F}(\mathfrak{h})$. Next, construct four maps:

$$\left\{ \begin{array}{lcl} \lambda_0(\bar{a}) & = & \lambda_{\varepsilon 0}(\pi_1 \bar{a}) + d^m \psi_1 \bar{a} - \psi_0 d\bar{a}, \\ \lambda_1(\bar{x}, \bar{y}) & = & \lambda_{\varepsilon 1}(\pi_0 \bar{x}, \pi_0 \bar{y}) - \psi_0[\bar{x}, \bar{y}] - [\psi_0 \bar{x}, \psi_0 \bar{y}] + G_0 \bar{x} \triangleright \psi_0 \bar{y} - G_0 \bar{y} \triangleright \psi_0 \bar{x}, \\ \lambda_2(\bar{x}, \bar{a}) & = & \lambda_{\varepsilon 2}(\pi_1 \bar{x}, \pi_1 \bar{a}) - \psi_1[\bar{x}, \bar{a}] - [\psi_0 \bar{x}, \psi_1 \bar{a}] + G_0 \bar{x} \triangleright \psi_1 \bar{a} - G_1 \bar{a} \triangleright \psi_0 \bar{x}, \\ \lambda_3(\bar{x}, \bar{y}, \bar{z}) & = & \lambda_{\varepsilon 3}(\pi_0 \bar{x}, \pi_0 \bar{y}, \pi_0 \bar{z}) - \psi_1 l_3(\bar{x}, \bar{y}, \bar{z}) + l_3^m(\psi_0 \bar{x}, \psi_0 \bar{y}, \psi_0 \bar{z}) \\ & & - (l_{\phi_0(G\bar{x})}(\psi_0 \bar{y}, \psi_0 \bar{z}) + (G_0 \bar{x}, G_0 \bar{y}) \triangleright \psi_0 \bar{z} + c.p.), \end{array} \right.$$

for any $\bar{x}, \bar{y}, \bar{z} \in \mathfrak{F}(\mathfrak{h})_0, \bar{a} \in \mathfrak{F}(\mathfrak{h})_1$. Since π, G and φ commute with d , we have

$$\begin{aligned}
\varphi_0(\lambda_0(\bar{a})) &= ds_1 \pi_1 \bar{a} - s_0 d\pi_1 \bar{a} + d(G_1 \bar{a} - s_1 \pi_1 \bar{a}) - (G_0 d\bar{a} - s_0 \pi_0 d\bar{a}) \\
&= 0.
\end{aligned}$$

Similarly, we can get $\varphi \circ \lambda_i = 0, i = 1, 2, 3$, that is, $\lambda = \sum_{i=0}^3 \lambda_i \in C^2(\mathfrak{F}(\mathfrak{h}), \mathbb{V})$.

Moreover, we claim that $D^{\mathfrak{F}(\mathfrak{h})} \lambda = \pi^* \theta_\varepsilon$. By straightforward calculations, we get

$$\begin{aligned}
(D^{\mathfrak{F}(\mathfrak{h})} \lambda)_0(\bar{x}, \bar{a}) &= \pi_0 \bar{x} \triangleright \lambda_0(\bar{a}) - \lambda_0[\bar{x}, \bar{a}] + \lambda_1(\bar{x}, d\bar{a}) - d^m \lambda_2(\bar{x}, \bar{a}) \\
&= \pi_0 \bar{x} \triangleright (\lambda_{\varepsilon 0}(\pi_1 \bar{a}) + d^m \psi_1 \bar{a} - \psi_0 d\bar{a}) - \lambda_{\varepsilon 0}(\pi_1 [\bar{x}, \bar{a}]) - d^m \psi_1 [\bar{x}, \bar{a}] + \psi_0 d[\bar{x}, \bar{a}] \\
&\quad + \lambda_{\varepsilon 1}(\pi_0 \bar{x}, \pi_0 d\bar{a}) - \psi_0[\bar{x}, d\bar{a}] - [\psi_0 \bar{x}, \psi_0 d\bar{a}] + G_0 \bar{x} \triangleright \psi_0 d\bar{a} - G_0 d\bar{a} \triangleright \psi_0 \bar{x} \\
&\quad - d^m (\lambda_{\varepsilon 2}(\pi_0 \bar{x}, \pi_1 \bar{a}) - \psi_1[\bar{x}, \bar{a}] - [\psi_0 \bar{x}, \psi_1 \bar{a}] + G_0 \bar{x} \triangleright \psi_1 \bar{a} - G_1 \bar{a} \triangleright \psi_0 \bar{x}) \\
&= s_0 \pi_0 \bar{x} \triangleright \lambda_{\varepsilon 0}(\pi_1 \bar{a}) - \lambda_{\varepsilon 0}(\pi_1 [\bar{x}, \bar{a}]) + \lambda_{\varepsilon 1}(\pi_0 \bar{x}, \pi_0 d\bar{a}) - d^m \lambda_{\varepsilon 2}(\pi_0 \bar{x}, \pi_1 \bar{a}) \\
&= \theta_{\varepsilon 0}(\pi_0 \bar{x}, \pi_1 \bar{a}) \\
&= (\pi^* \theta_{\varepsilon 0})(\bar{x}, \bar{a}),
\end{aligned}$$

where we have used condition (i) of Definition 3.2. Likewise, we deduce that $D^{\mathfrak{F}(\mathfrak{h})} \lambda = \pi^* \theta_\varepsilon$.

So we can use such a defined λ to construct a crossed module ε_λ as in Proposition 5.2. In the following, we prove there is a map (F, G) from ε_λ to ε in $\mathcal{C}(\mathfrak{h}, \mathbb{V})$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{V} & \xrightarrow{i} & \ker \pi \oplus_\lambda \mathbb{V} & \xrightarrow{\varphi^\lambda} & \mathfrak{F}(\mathfrak{h}) & \xrightarrow{\pi} & \mathfrak{h} & \longrightarrow & 0 \\
& & Id \downarrow & & F \downarrow & & G \downarrow & & Id \downarrow & & \\
0 & \longrightarrow & \mathbb{V} & \xrightarrow{i'} & \mathfrak{m} & \xrightarrow{\varphi} & \mathfrak{g} & \xrightarrow{\pi'} & \mathfrak{h} & \longrightarrow & 0,
\end{array}$$

where $F(\alpha + u) = i'(u) + qG(\alpha)$, $\forall \alpha \in \ker \pi, u \in \mathbb{V}$. Indeed, we shall prove F is a strong Lie 2-algebra homomorphism compatible with the actions, that is, for any $\alpha \in \ker \pi, u \in \mathbb{V}, \bar{x} \in \mathfrak{F}(\mathfrak{h})$,

$$F(\bar{x} \triangleright_\lambda (\alpha + u)) = G(\bar{x}) \triangleright F(\alpha + u), \quad F((\bar{x}, \bar{y}) \triangleright_\lambda (\alpha + u)) = (G\bar{x}, G\bar{y}) \triangleright F(\alpha + u).$$

We only sketch the proof of compatibility. For any $\alpha \in \ker \pi_0, u \in V_0$ and $\bar{x}_0 \in \mathfrak{F}(\mathfrak{h})_0$, we have

$$\begin{aligned} F_0(\bar{x} \triangleright_\lambda (\alpha + u)) &= F_0([\bar{x}, \alpha] + \lambda_1(\bar{x}, \alpha) + \pi_0 \bar{x} \triangleright u) \\ &= \lambda_1(\bar{x}, \alpha) + \pi_0 \bar{x} \triangleright i'_0 u + q_0 G_0[\bar{x}, \alpha] \\ &= -q_0 G_0[\bar{x}, \alpha] - [q_0 G_0 \bar{x} - q_0 s_0 \pi_0 \bar{x}, q_0 G_0 \alpha] + G_0 \bar{x} \triangleright q_0 G_0 \alpha \\ &\quad - G_0 \alpha \triangleright (q_0 G_0 \bar{x} - q_0 s_0 \pi_0 \bar{x}) + \pi_0 \bar{x} \triangleright i'_0 u + q_0 G_0[\bar{x}, \alpha] \\ &= \pi_0 \bar{x} \triangleright i'_0 u + G_0 \bar{x} \triangleright q_0 G_0 \alpha \\ &= G_0 \bar{x} \triangleright F_0(\alpha + u), \end{aligned}$$

where we have used the condition (i) of Definition 3.2 and $\pi_0 \bar{x} \triangleright i'_0 u = \pi'_0 G_0 \bar{x} \triangleright i'_0 u = G_0 \bar{x} \triangleright i'_0 u$. Next, since $(\pi_0 \bar{x}, \pi_0 \bar{y}) \triangleright i'_0 u = (G_0 \bar{x}, G_0 \bar{y}) \triangleright i'_0 u$ due to the fixed action of \mathfrak{h} on \mathbb{V} , we have

$$\begin{aligned} &F_1((\bar{x}, \bar{y}) \triangleright_\lambda (\alpha + u)) \\ &= F_1(l_3(\bar{x}, \bar{y}, \alpha) + \lambda_3(\bar{x}, \bar{y}, \alpha) + (\pi_0 \bar{x}, \pi_0 \bar{y}) \triangleright u) \\ &= -q_1 G_1 l_3(\bar{x}, \bar{y}, \alpha) + l_3^{\mathfrak{m}}(q_0 G_0 \bar{x} - q_0 s_0 \pi_0 \bar{x}, q_0 G_0 \bar{y} - q_0 s_0 \pi_0 \bar{y}, q_0 G_0 \alpha) \\ &\quad - (G_0 \bar{x}, G_0 \bar{y}) \triangleright q_0 G_0 \alpha - (G_0 \bar{y}, G_0 \alpha) \triangleright (q_0 G_0 \bar{x} - q_0 s_0 \pi_0 \bar{x}) - (G_0 \bar{x}, G_0 \alpha) \triangleright (q_0 G_0 \bar{y} - q_0 s_0 \pi_0 \bar{y}) \\ &\quad + (G_0 \bar{x}, G_0 \bar{y} - s_0 \pi_0 \bar{y}) \triangleright q_0 G_0 \alpha + (G_0 \bar{y}, G_0 \alpha) \triangleright (q_0 G_0 \bar{x} - q_0 s_0 \pi_0 \bar{x}) \\ &\quad + (G_0 \alpha, G_0 \bar{x} - s_0 \pi_0 \bar{x}) \triangleright (q_0 G_0 \bar{y} - q_0 s_0 \pi_0 \bar{y}) + q_1 G_1 l_3(\bar{x}, \bar{y}, \alpha) + (\pi_0 \bar{x}, \pi_0 \bar{y}) \triangleright i'_0 u \\ &= (\pi_0 \bar{x}, \pi_0 \bar{y}) \triangleright i'_0 u - (G_0 \bar{x}, G_0 \bar{y}) \triangleright q_0 G_0 \alpha \\ &= (G_0 \bar{x}, G_0 \bar{y}) \triangleright F_0(\alpha + u), \end{aligned}$$

where we have used condition (i), (ii), (iii) of Definition 3.2. The other hand is a consequence of Lemma 5.13 and 5.14. This ends the proof. ■

Next, we give an alternative description of the equivalence relation in Theorem 5.10, which is similar to the statements in [24, 4]. Two strong crossed modules $(\mathfrak{m}, \mathfrak{g}, \phi, \varphi), (\mathfrak{m}', \mathfrak{g}', \phi', \varphi') \in \mathcal{C}(\mathfrak{h}, \mathbb{V})$ are called **elementary equivalent** if there is a morphism of crossed modules (F, G, τ) such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{V} & \xrightarrow{i} & \mathfrak{m} & \xrightarrow{\varphi} & \mathfrak{g} & \xrightarrow{\pi} & \mathfrak{h} & \longrightarrow & 0 \\ & & Id \downarrow & & F \downarrow & & G \downarrow & & Id \downarrow & & \\ 0 & \longrightarrow & \mathbb{V} & \xrightarrow{i'} & \mathfrak{m}' & \xrightarrow{\varphi'} & \mathfrak{g}' & \xrightarrow{\pi'} & \mathfrak{h} & \longrightarrow & 0, \end{array}$$

is commutative and

$$G_2 = 0; \quad \text{Im} \tau \subset i'(\mathbb{V}); \quad \tau(\mathfrak{g}_0 \wedge i(\mathbb{V})) = 0; \quad \tau(\varphi_0 \alpha, \beta) = \tau(\alpha, \varphi_0 \beta), \quad \forall \alpha, \beta \in \mathfrak{m}_0.$$

By a straightforward but tedious computation combined with the proof of Theorem 5.10, we obtain the following proposition.

Proposition 5.17. *The equivalence relation generated by elementary equivalence relation coincides with the equivalence relation \sim in Theorem 5.10.*

One key step in the proof of Theorem 5.10 is the construction of the map $\mu : \mathcal{C}(\mathfrak{h}, \mathbb{V}) \rightarrow H^3(\mathfrak{h}, \mathbb{V})$. The following example illustrates that for a particular class of crossed modules given in Example 3.10, the map μ is closely related to the connecting map in the long exact sequence of cohomology groups.

Example 5.18. Consider the strong crossed module $\varepsilon = (\mathbb{I}, \mathfrak{h} \oplus_\lambda \mathbb{Q}, \phi, \varphi) \in \mathcal{C}(\mathfrak{h}, \mathbb{V})$ obtained in Example 3.10. The sequence (15) induces a short sequence of complexes

$$0 \rightarrow C^*(\mathfrak{h}, \mathbb{V}) \xrightarrow{\bar{p}} C^*(\mathfrak{h}, \mathbb{I}) \xrightarrow{\bar{q}} C^*(\mathfrak{h}, \mathbb{Q}) \rightarrow 0.$$

Since for arbitrary \mathfrak{h} -module homomorphism f , we have $\bar{f} \circ D^{\mathfrak{h}} = D^{\mathfrak{h}} \circ \bar{f}$. Thus p, q induce maps between cohomology groups. Moreover, similar to the process in homological algebra, we can construct a connecting homomorphism $\partial : H^*(\mathfrak{h}, \mathbb{Q}) \rightarrow H^{*+1}(\mathfrak{h}, \mathbb{V})$ such that

$$\dots \rightarrow H^*(\mathfrak{h}, \mathbb{V}) \xrightarrow{\bar{p}} H^*(\mathfrak{h}, \mathbb{I}) \xrightarrow{\bar{q}} H^*(\mathfrak{h}, \mathbb{Q}) \xrightarrow{\partial} H^{*+1}(\mathfrak{h}, \mathbb{V}) \rightarrow \dots$$

is a long exact sequence of cohomology groups.

In particular, we have $\mu(\varepsilon) = \partial[\lambda]$. See [24, Theorem 3] for more details.

References

- [1] J.C. Baez and A.S. Crans, Higher-dimensional algebra VI: Lie 2-algebras, *Theory Appl. Categ.*, 2004, 12: 492-538.
- [2] C.M. Bai, Y.H. Sheng and C.C. Zhu, Lie 2-bialgebra, *Comm. Math. Phys.*, 2013, 320(1): 149-172.
- [3] H.J. Baues and E.G. Minian, Crossed extensions of algebras and Hochschild cohomology, *Homol. Homot. Appl.*, 2002, 4(2): 63-82.
- [4] J.M. Casas, E. Khmaladze and M. Ladra, Crossed modules for Leibniz n -algebras, *Forum Math.*, 2008, 20(5): 841-858.
- [5] S. Chen, Y. Sheng and Z. Zheng, Non-abelian extensions of Lie 2-algebras, *Sci. China Math.*, 2012, 55(8): 1655-1668.
- [6] Z. Chen, M. Stinson and P. Xu, Weak Lie 2-bialgebras, *J. Geom. Phys.*, 2013, 68: 59-68.
- [7] D. Conduché, Modules croisés generalisés de Longueur 2, *J. Pure Appl. Alg.* 1984, 34: 155-178.
- [8] G.J. Ellis, Homotopical aspects of Lie algebras, *J. Austral. Math. Soc., Ser. A*, 1993, 54(3): 393-419.
- [9] J. Faria Martins and R. Picken, The fundamental Gray 3-groupoid of a smooth manifold and local 3-dimensional holonomy based on a 2-crossed module, *Diff. Geom. Appl.*, 2011, 29(2): 179-206.
- [10] M. Gerstenhaber, The cohomology structure of an associate ring, *Ann. Math.*, 1963, 78(2): 267-288.
- [11] M. Gerstenhaber, A uniform cohomology theory for algebras, *Pros. Nat. Acad. Sci. U.S.A.*, 1964, 51: 626-629.
- [12] G. Ginot and P. Xu, Cohomology of Lie 2-groups, *L'Enseignement Mathématique*, 2009, 55(3): 373-396.
- [13] J.L. Loday, Spaces having finitely many non-trivial homotopy groups, *J. Pure Appl. Alg.*, 1982, 24: 179-202.

- [14] T. Lada and M. Markl, Strongly homotopy Lie algebras, *Comm. Alg.* 1995, 23(6): 2147-2161.
- [15] Z.J. Liu, Y.H. Sheng and T. Zhang, Deformations of Lie 2-algebras, arXiv: 1306.6225.
- [16] M. Markl, A cohomology theory for A (m)-algebras and applications, *J. Pure Appl. Alg.*, 1992, 83(2): 141-175.
- [17] M. Markl, Free homotopy algebras, *Homol. Homot. Appl.*, 2005, 7(2): 123-137.
- [18] J. Millès, André-Quillen cohomology of algebras over an operad, *Adv. Math.*, 2011, 226(6): 5120-5164.
- [19] I. Moerdijk, Orbifolds as groupoids: an introduction, Orbifolds in mathematics and physics (Madison, WI, 2001), *Contemp. Math.*, 2002, 310, 205C222.
- [20] A. Mutlu and T. Porter, Crossed squares and 2-crossed modules, arXiv: 0210462.
- [21] K.L. Norrie, Actions and automorphisms of crossed modules, *Bull. Soc. Math. France*, 1990, 118(2): 129-146.
- [22] M. Penkava, L-infinity algebras and their cohomology, arXiv: 9512014.
- [23] M. Schlessinger and J. Stasheff, The Lie algebra structure of tangent cohomology and deformation theory, *J. Pure Appl. Alg.* 1985, 38: 313-322.
- [24] F. Wagemann, On Lie algebra crossed modules, *Comm. Alg.*, 2006, 34(5): 1699-1722.